

Convergence to travelling waves in the Fisher-Kolmogorov equation with a non-Lipschitzian reaction term

Pavel DRÁBEK

Department of Mathematics and
N.T.I.S. (Center of New Technologies for Information Society)
University of West Bohemia
P.O. Box 314
CZ-306 14 Plzeň, Czech Republic
e-mail: pdrabek@kma.zcu.cz

and

Peter TAKÁČ
Institut für Mathematik
Universität Rostock
Ulmenstraße 69, Haus 3
D-18055 Rostock, Germany
e-mail: peter.takac@uni-rostock.de

May 19, 2016

ABSTRACT. We consider the semilinear Fisher-Kolmogorov-Petrovski-Piscounov equation for the advance of an advantageous gene in biology. Its nonsmooth reaction function $f(u)$ allows for the introduction of travelling waves with a new profile. We study existence, uniqueness, and long-time asymptotic behavior of the solutions $u(x, t)$, $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. We prove also the existence and uniqueness (up to a spatial shift) of a travelling wave U . Our main result is the uniform convergence (for $x \in \mathbb{R}$) of every solution $u(x, t)$ of the Cauchy problem to a single travelling wave $U(x - ct + \zeta)$ as $t \rightarrow \infty$. The speed c and the travelling wave U are determined uniquely by f , whereas the shift ζ is determined by the initial data.

Running head: Convergence to a travelling wave in the FKPP equation

Keywords: Fisher-Kolmogorov equation, nonsmooth reaction function,
travelling waves, solutions of the Cauchy problem,
long-time behavior

2010 Mathematics Subject Classification: Primary 35Q92, 35K91;
Secondary 35K58, 92B05.

1 Introduction

We are concerned with the long-time asymptotic behavior of solutions to the Cauchy problem for the favorite *Fisher-KPP equation* (or *Fisher-Kolmogorov equation*):

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(u) & \text{for } (x, t) \in \mathbb{R} \times \mathbb{R}_+; \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

This equation was derived by R. A. FISHER [8] in 1937 and first mathematically analyzed by A. KOLMOGOROV, I. PETROVSKI, and N. PISCOUNOV [12] in the same year. This is a mathematical problem originating in a morphogenesis model described in J. D. MURRAY [16], §13.3, pp. 444–449. More precisely, we are interested in the convergence of a solution $u(x, t)$ to a *travelling wave* $U(x - ct)$ as $t \rightarrow +\infty$, uniformly for $x \in \mathbb{R}$. Of course, $c \in \mathbb{R}$ stands for the *speed* of the travelling wave. In particular, we investigate the *stability* of travelling waves. We assume $f(0) = f(1) = 0$ and focus on biologically (or physically) meaningful solutions satisfying $0 \leq u(x, t) \leq 1$ as u corresponds to the ratio of an advantageous gene in a population. Obviously, the “extreme” cases $u \equiv 0$ or $u \equiv 1$ throughout $\mathbb{R} \times \mathbb{R}_+$ are trivial solutions. Consequently, we will be interested primarily in nontrivial solutions $0 \leq u(x, t) \leq 1$, with $u \not\equiv 0$ and $u \not\equiv 1$, and in nontrivial travelling waves $0 \leq U(x - ct) \leq 1$ connecting the two trivial equilibrium states, i.e.,

$$(1.2) \quad \lim_{z \rightarrow -\infty} U(z) = 0 \quad \text{and} \quad \lim_{z \rightarrow +\infty} U(z) = 1.$$

The long-time asymptotic behavior of the dynamical system generated by the Fisher-KPP equation (1.1) is studied, e.g., in P. C. FIFE and J. B. MCLEOD [7] under the standard hypothesis of $f : \mathbb{R} \rightarrow \mathbb{R}$ continuously differentiable (i.e., of class C^1). This crucial hypothesis guarantees not only the existence of a unique classical solution $u : \mathbb{R} \times \mathbb{R}_+ \rightarrow [0, 1]$ of the Cauchy (initial value) problem (1.1), but facilitates also the linearization of this problem about a travelling wave $U(x - ct)$. Such solutions generate a “smooth” (C^1) dynamical system whose properties can be investigated by well-known methods; see, e.g., J. K. HALE [9] or D. HENRY [11]. Further studies of convergence of solutions to a travelling wave and front propagation in the semilinear Fisher-KPP equation with a C^1 -reaction function f can be found in D. G. ARONSON and H. F. WEINBERGER [1], F. HAMEL and N. NADIRASHVILI [10], and H. MATANO and T. OGIWARA [14, 15]. Unlike in [14, 15], we do not impose any stability condition on a travelling wave $U : \mathbb{R} \rightarrow \mathbb{R}$, nor do we assume that f is C^1 , to obtain its monotonicity. Thanks to our “global” conditions on f , we are able to prove the monotonicity of U directly from the equation.

In our present work we relax the differentiability hypothesis on f to being only α -Hölder-continuous ($0 < \alpha < 1$) and “one-sided” Lipschitz-continuous (i.e., $s \mapsto f(s) - Ls : \mathbb{R} \rightarrow \mathbb{R}$ is monotone decreasing, for some constant $L \in \mathbb{R}_+$). In particular, our hypotheses allow for the singular derivatives

$$(1.3) \quad f'(0) = \lim_{s \rightarrow 0} \frac{f(s)}{s} = -\infty \quad \text{and} \quad f'(1) = \lim_{s \rightarrow 1} \frac{f(s)}{s - 1} = -\infty.$$

This type of a reaction function f has been studied extensively in biological models of various kinds of *logistic growth* in A. TSOULARIS and J. WALLACE [19].

The situation sketched in (1.3) excludes application of the standard linearization procedure about a travelling wave widely used in [7, 14, 15, 16]. Our weaker differentiability conditions on f allow for a wider and, perhaps, also more realistic class of travelling waves. In contrast to the usual type of travelling waves $U(x - ct)$ that appear in [7, 16] with $U'(z) > 0$ for all $z \in \mathbb{R}$ and the limits (1.2), we allow also for

$$(1.4) \quad \begin{cases} U(z) = 0 & \text{if } -\infty < z \leq z_0; \\ U'(z) > 0 & \text{if } z_0 < z < z_1; \\ U(z) = 1 & \text{if } z_1 \leq z < \infty, \end{cases}$$

for some $-\infty < z_0 < z_1 < \infty$. Mathematically, this phenomenon may occur due to $f(s)$ not being Lipschitz-continuous at the equilibrium points $s = 0$ and $s = 1$; cf. (1.3) and Example 2.4. Since the linearization procedure is not available, we are forced to develop alternative methods to treat the dynamical system generated by (1.1) and to investigate its long-time asymptotic behavior. Whereas the uniqueness of a solution $u : \mathbb{R} \times \mathbb{R}_+ \rightarrow [0, 1]$ to the Cauchy problem (1.1) is guaranteed by the hypothesis on “one-sided” Lipschitz continuity of f , the family of all possible travelling waves has to be retrieved by entirely different methods partially developed already in P. DRÁBEK, R. F. MANÁSEVICH, and P. TAKÁČ [4] and P. DRÁBEK and P. TAKÁČ [6] for f lacking the Lipschitz continuity. The uniqueness of their speed c is proved in [6, Theorem 3.1], whereas the monotonicity and uniqueness (up to a spatial translation) of their profile U will be proved in our present article (see Proposition 2.3, Part (b)).

Before stating our main convergence result below, we would like to remark that our hypotheses on the reaction function $f : \mathbb{R} \rightarrow \mathbb{R}$ stated in Sections 2 and 3, (H1) and (H2), guarantee the uniqueness of the speed $c \in \mathbb{R} \setminus \{0\}$ and the profile $U : \mathbb{R} \rightarrow \mathbb{R}$ (up to a spatial translation); see Proposition 2.3, Parts (b) and (d). Adding another hypothesis on f in §3.2, (H3), we will prove in Proposition 3.3 that the Cauchy problem (1.1) possesses a unique mild solution $u : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ for any initial data $u_0 \in L^\infty(\mathbb{R})$ satisfying $0 \leq u_0 \leq 1$ in \mathbb{R} . Our main result can be stated loosely as follows:

Theorem 1.1 (Convergence to a travelling wave.) *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be Hölder-continuous and $f(0) = f(s_0) = f(1) = 0$ for some $0 < s_0 < 1$. Assume that f satisfies some additional “technical” hypotheses that guarantee also $-\infty < z_0 < z_1 < +\infty$. Finally, assume that the initial data $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ are Lebesgue-measurable and satisfy $0 \leq u_0(x) \leq 1$ for every $x \in \mathbb{R}$, together with*

$$\limsup_{x \rightarrow -\infty} u_0(x) < s_0 < \liminf_{x \rightarrow +\infty} u_0(x),$$

meaning

$$\lim_{n \rightarrow \infty} \operatorname{ess\,sup}_{-\infty < x \leq -n} u_0(x) < s_0 < \lim_{n \rightarrow \infty} \operatorname{ess\,inf}_{n \leq x < +\infty} u_0(x).$$

Then the (unique mild) solution $u : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ of problem (1.1) satisfies $0 \leq u(x, t) \leq 1$ for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ and there exists a unique spatial shift $\zeta \in \mathbb{R}$ such that

$$(1.5) \quad \sup_{x \in \mathbb{R}} |u(x, t) - U(x - ct + \zeta)| \longrightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This theorem is easily derived from Theorem 4.6 proved in Section 4 using the transformation with the moving coordinate $z = x - ct$.

This article is organized as follows. In the next section (Section 2) we analyze the travelling waves. The existence and uniqueness of a solution to the Cauchy (initial value) problem (1.1) are established in Section 3. In this section we prove also the usual weak comparison principle for parabolic problems (§3.2). Moreover, we construct a special pair of ordered sub- and supersolutions to the Cauchy problem (1.1) by modifying the travelling waves (§3.4). This pair is subsequently used in Section 4 to prove the Lyapunov stability of a travelling wave (§4.1) as well as the convergence of a solution $u(x, t)$ to a single travelling wave $U(x - ct + \zeta)$ as $t \rightarrow \infty$, uniformly for $x \in \mathbb{R}$; cf. Theorem 4.6.

2 Analysis of Travelling Waves

We reformulate the Cauchy problem (1.1) for $u(x, t)$ as an equivalent initial value problem for the unknown function $v(z, t) = u(z + ct, t) \equiv u(x, t)$ with the moving coordinate $z = x - ct$:

$$(2.1) \quad \begin{cases} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial z^2} - c \frac{\partial v}{\partial z} = f(v), & (z, t) \in \mathbb{R} \times \mathbb{R}_+; \\ v(z, 0) = v_0(z). \end{cases}$$

We will show in Proposition 2.1 (§2.1 below) that every travelling wave $u(x, t) = U(x - ct)$ for (1.1) must have a monotone increasing profile $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfying (1.2) and $U'(z) > 0$ for every $z \in \mathbb{R}$ such that $0 < U(z) < 1$. Hence, by [6, Theorem 3.1], the speed c of a travelling wave $U(x - ct)$ in eq. (1.1) is determined uniquely by f . This means that we are able to “reduce” the problem of finding all *travelling waves* $u(x, t) = U(x - ct)$ for (1.1) to the problem of investigating all *stationary solutions* $v(z, t) = U(z)$ for (2.1) that obey also (1.2). Of course, $v_0 \equiv u_0$ in \mathbb{R} . Since both these families are described by the same profile $U : \mathbb{R} \rightarrow [0, 1]$ that obeys (1.2), the uniqueness of which (up to a spatial translation) will be proved later in Proposition 2.3, Part (b), we introduce the following terminology for the latter: We call $v(z, t) = U(z)$ and any other stationary solution $v(z, t) = U(z + \zeta)$, $\zeta \in \mathbb{R}$ – a constant, that obeys (1.2), a **TW-solution** of problem (2.1). It should not to be confused with the travelling wave $u(x, t) = U(x - ct + \zeta)$ in the original problem (1.1). In fact, to identify all possible TW-solutions of problem (2.1), (1.2), we first establish their monotonicity (in Proposition 2.1), then prove the uniqueness (up to a spatial translation) of their profile $U : \mathbb{R} \rightarrow [0, 1]$ satisfying (1.2) (in Proposition 2.3, Part (b)). Finally, all possible TW-solutions are described in Proposition 2.3, Part (d). Thus, except when we wish to stress the biological (or physical) meaning of travelling waves, we prefer to treat the Cauchy problem (2.1) and investigate its TW-solutions, rather than problem (1.1).

As usual, we denote $\mathbb{R} \stackrel{\text{def}}{=} (-\infty, \infty)$, $\mathbb{R}_+ \stackrel{\text{def}}{=} [0, \infty)$, and assume that the reaction term f satisfies the following basic hypotheses:

(H1) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, but *not necessarily smooth* function, such that $f(0) =$

$f(s_0) = f(1) = 0$ for some $0 < s_0 < 1$, together with $f(s) < 0$ for every $s \in (0, s_0) \cup (1, \infty)$, $f(s) > 0$ for every $s \in (-\infty, 0) \cup (s_0, 1)$, and

$$(2.2) \quad F(r) \stackrel{\text{def}}{=} \int_0^r f(s) \, ds < 0 \quad \text{whenever } 0 < r \leq 1.$$

We remark that the case $F(1) = 0$ would prevent the existence of any travelling wave with speed $c \neq 0$; cf. eq. (2.7). That is why we assume $F(1) < 0$. We will see in our proof of Proposition 2.3 (in §2.2 below) that Hypothesis **(H1)** is sufficient to determine the speed c and the profile $U : \mathbb{R} \rightarrow [0, 1]$ uniquely, the latter up to a spatial translation. The reaction function f satisfying **(H1)** models the so-called *heterozygote inferior* case in the genetic model studied, e.g., in D. G. ARONSON and H. F. WEINBERGER [1, p. 35].

In the next paragraph, §2.1, we will show that $U : \mathbb{R} \rightarrow \mathbb{R}$ must be monotone increasing with $U' > 0$ on a suitable open interval $(z_0, z_1) \subset \mathbb{R}$, such that

$$(2.3) \quad \lim_{z \rightarrow z_0+} U(z) = 0 \quad \text{and} \quad \lim_{z \rightarrow z_1-} U(z) = 1.$$

We would like to remark that the cases of $z_0 > -\infty$ and/or $z_1 < +\infty$ render qualitatively different travelling waves than the classical case $(z_0, z_1) = \mathbb{R}$ which has been studied in the original works [8, 12] and in the literature ([1, 7, 10, 16, 17]). Using this setting, in paragraph §2.2 we are able to find a *first integral* for the second-order equation for U :

$$(2.4) \quad \frac{d^2 U}{dz^2} + c \frac{dU}{dz} + f(U) = 0, \quad z \in (z_0, z_1).$$

2.1 Monotonicity of a Travelling Wave

In accordance with problem (2.4), (1.2), we investigate the following problem for an unknown C^2 -function $v : \mathbb{R} \rightarrow \mathbb{R}$:

$$(2.5) \quad v''(z) + c v'(z) + f(v(z)) = 0, \quad z \in \mathbb{R};$$

$$(2.6) \quad \lim_{z \rightarrow -\infty} v(z) = 0 \quad \text{and} \quad \lim_{z \rightarrow +\infty} v(z) = 1.$$

We would like to stress that here we do not impose any hypothesis on the reaction function $f : \mathbb{R} \rightarrow \mathbb{R}$ that would require some kind of Hölder or Lipschitz continuity, not even one-sided. Thus, we cannot use standard “local” uniqueness arguments for eq. (2.5) to find the family of all C^2 -solutions $v : \mathbb{R} \rightarrow \mathbb{R}$ to problem (2.5), (2.6). Nevertheless, we will be able to demonstrate that, if the constant $c \in \mathbb{R}$ is such that at least one solution U to this problem exists, then any other solution $v : \mathbb{R} \rightarrow \mathbb{R}$ can be obtained by a translation in the argument of U by a fixed number $\zeta \in \mathbb{R}$, i.e., $v(z) = U(z + \zeta)$ for $z \in \mathbb{R}$; cf. Proposition 2.3 (§2.2) below. In order to treat questions related to the uniqueness of a solution v , we take advantage of the global properties of v , such as the (strict) monotonicity established in the next proposition, that are consequences of the global behavior of the reaction function $f : \mathbb{R} \rightarrow \mathbb{R}$ specified in Hypothesis **(H1)**.

Before beginning to investigate rather subtle properties of travelling waves, we recall the following well-known identity which holds for any solution $v : \mathbb{R} \rightarrow \mathbb{R}$ to problem (2.5), (2.6):

$$(2.7) \quad c \int_{-\infty}^{+\infty} v'(z)^2 dz = F(0) - F(1) = -F(1) \quad (> 0).$$

Consequently, $c > 0$. This identity is obtained by calculating the first integral for (2.5),

$$(2.8) \quad \frac{1}{2} \frac{d}{dz} (v'(z)^2) + c v'(z)^2 + \frac{d}{dz} F(v(z)) = 0, \quad z \in \mathbb{R},$$

integrating it over a suitable sequence of intervals $(x_k, y_k) \subset \mathbb{R}$; $k = 1, 2, 3, \dots$, with $-x_k, y_k \in (k, k+1)$,

$$(2.9) \quad \frac{1}{2} (v'(y_k)^2) - \frac{1}{2} (v'(x_k)^2) + c \int_{x_k}^{y_k} v'(z)^2 dz + F(v(y_k)) - F(v(x_k)) = 0,$$

and letting $k \rightarrow \infty$. Here, by the mean value theorem, we construct $x_k \in (-k-1, -k)$ and $y_k \in (k, k+1)$ such that $v'(x_k) = v(-k) - v(-k-1)$ and $v'(y_k) = v(k+1) - v(k)$, respectively. Thanks to the limits in (2.6), we arrive at $\lim_{k \rightarrow \infty} v'(x_k) = \lim_{k \rightarrow \infty} v'(y_k) = 0$. Hence, (2.7) follows from (2.6) and (2.9), by letting $k \rightarrow \infty$.

We have the following result.

Proposition 2.1 (Monotonicity.) *Let $c \in (0, \infty)$ and assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies Hypothesis (H1). Then every C^2 -solution $v : \mathbb{R} \rightarrow \mathbb{R}$ of eq. (2.5) with the limits (2.6) satisfies $0 \leq v(z) \leq 1$ and $v'(z) \geq 0$ for all $z \in \mathbb{R}$. Moreover, there is an open interval $(z_0, z_1) \subset \mathbb{R}$, $-\infty \leq z_0 < z_1 \leq +\infty$, such that $v' > 0$ on (z_0, z_1) together with*

$$(2.10) \quad \begin{cases} \lim_{z \rightarrow z_0+} v(z) = 0 & \text{and} & v(z) = 0 \text{ if } -\infty < z \leq z_0, \\ \lim_{z \rightarrow z_1-} v(z) = 1 & \text{and} & v(z) = 1 \text{ if } z_1 \leq z < +\infty. \end{cases}$$

Proof. We verify our first claim, $0 \leq v(z) \leq 1$ for all $z \in \mathbb{R}$, by contradiction. Suppose there is a number $\xi \in \mathbb{R}$ such that $v(\xi) < 0$. We make use of the limits in (2.6) to conclude that there are numbers $\xi_1, \xi_2 \in \mathbb{R}$ such that $\xi_1 < \xi < \xi_2$ and $v(\xi) < \min\{v(\xi_1), v(\xi_2)\}$. Denoting by $\xi_0 \in [\xi_1, \xi_2]$ the (global) minimizer for the function v over the compact interval $[\xi_1, \xi_2]$, we arrive at $\xi_0 \in (\xi_1, \xi_2)$, $v(\xi_0) \leq v(\xi) < 0$, $v'(\xi_0) = 0$, and $v''(\xi_0) \geq 0$. Hence, we have also $f(v(\xi_0)) > 0$, by Hypothesis (H1). We insert these properties of $v(z)$ at $z = \xi_0$ into the left-hand side of eq. (2.5), thus arriving at

$$v''(\xi_0) + c v'(\xi_0) + f(v(\xi_0)) > 0,$$

a contradiction to eq. (2.5).

We proceed analogously if $\xi \in \mathbb{R}$ is such that $v(\xi) > 1$. The limits in (2.6) guarantee that there are numbers $\xi_1, \xi_2 \in \mathbb{R}$ such that $\xi_1 < \xi < \xi_2$ and $v(\xi) > \max\{v(\xi_1), v(\xi_2)\}$.

Denoting by $\xi_0 \in [\xi_1, \xi_2]$ the maximizer for the function v over the compact interval $[\xi_1, \xi_2]$, we arrive at $\xi_0 \in (\xi_1, \xi_2)$, $v(\xi_0) \geq v(\xi) > 1$, $v'(\xi_0) = 0$, and $v''(\xi_0) \leq 0$. Hence, we have also $f(v(\xi_0)) < 0$, by Hypothesis **(H1)**. Similarly as above, we arrive at

$$v''(\xi_0) + c v'(\xi_0) + f(v(\xi_0)) < 0,$$

a contradiction to eq. (2.5) again. We have verified that $0 \leq v(z) \leq 1$ holds for all $z \in \mathbb{R}$.

Now it follows from (2.6) again that there is a maximal (nonempty) open interval $(z_0, z_1) \subset \mathbb{R}$ such that $v'(z) > 0$ holds for all $z \in (z_0, z_1)$. The strict monotonicity of v in (z_0, z_1) guarantees the existence of the (monotone) limits

$$(2.11) \quad 0 \leq \ell_0 \stackrel{\text{def}}{=} \lim_{z \rightarrow z_0+} v(z) < \ell_1 \stackrel{\text{def}}{=} \lim_{z \rightarrow z_1-} v(z) \leq 1.$$

The maximality of the interval (z_0, z_1) forces either $(z_0, z_1) = \mathbb{R}$ in which case the proposition is proved, or else at least one of the inequalities $z_0 > -\infty$ and $z_1 < +\infty$ must be valid: If $z_0 > -\infty$ then $v'(z_0) = 0$, whereas if $z_1 < +\infty$ then $v'(z_1) = 0$.

We begin to investigate these two possibilities for the following two extremal cases:

- (i) $z_0 > -\infty$, $v'(z_0) = 0$, and $v(z_0) = 0$;
- (ii) $z_1 < +\infty$, $v'(z_1) = 0$, and $v(z_1) = 1$.

To treat these cases, we introduce the Lyapunov function $V \equiv V[v] : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(2.12) \quad V(z) \equiv V[v](z) \stackrel{\text{def}}{=} \frac{1}{2} [v'(z)]^2 + F(v(z)), \quad z \in \mathbb{R}.$$

Taking advantage of eq. (2.5), we calculate

$$(2.13) \quad \frac{d}{dz} V(z) = v' v'' + f(v(z)) v' = -c [v'(z)]^2 \leq 0 \quad \text{for all } z \in \mathbb{R}.$$

In particular, if $V(\zeta_1) \leq V(\zeta_2)$ for some $-\infty < \zeta_1 < \zeta_2 < +\infty$, then we have $v'(z) \equiv 0$ and $v(z) \equiv v(\zeta_1) = v(\zeta_2)$ for every $z \in [\zeta_1, \zeta_2]$.

Case (i). We claim that $v(z_0) = 0$ implies $v(z) = 0$ for all $z \leq z_0$. *Proof:* Since $0 \leq v(z) \leq 1$ holds for all $z \in \mathbb{R}$, we have also $v'(z_0) = 0$. Thus, on the contrary to our claim, suppose that $v(\hat{z}) > 0$ at some $\hat{z} \in (-\infty, z_0)$. Taking into account also $\lim_{z \rightarrow -\infty} v(z) = 0$, we conclude that the function v attains its (global) maximum $v(\xi)$ over the interval $(-\infty, z_0]$ at some point $\xi \in (-\infty, z_0)$. Consequently, we have $v'(\xi) = 0$ and $v(z_0) = 0 < v(\hat{z}) \leq v(\xi) \leq 1$. We make use of standard properties of the Lyapunov function V to compare

$$F(v(\xi)) = V(\xi) > V(z_0) = F(v(z_0)) = F(0) = 0$$

which contradicts our hypothesis (2.2), i.e., $F(r) < 0$ with $r = v(\xi) \in (0, 1]$. We have proved that $v(z) = 0$ holds for all $z \leq z_0$.

Case (ii). We claim that $v(z_1) = 1$ implies $v(z) = 1$ for all $z \geq z_1$. *Proof:* Again, from $0 \leq v \leq 1$ we deduce also $v'(z_1) = 0$. Thus, on the contrary to our claim, suppose that $v(\hat{z}) < 1$ at some $\hat{z} \in (z_1, +\infty)$. Then standard properties of the Lyapunov function V yield

$$0 > F(1) = F(v(z_1)) = V(z_1) > V(\hat{z}) \geq V(z) \geq F(v(z)) \quad \text{for all } z \geq \hat{z}.$$

Letting $z \rightarrow +\infty$ and using $\lim_{z \rightarrow +\infty} v(z) = 1$, we arrive at

$$F(1) > V(\hat{z}) \geq \lim_{z \rightarrow +\infty} F(v(z)) = F(1),$$

a contradiction again. We have verified $v(z) = 1$ for all $z \geq z_1$.

In our next step we show that there is no number $\xi \in \mathbb{R}$ such that $v'(\xi) = 0$ and $0 < v(\xi) < 1$. On the contrary, if $\xi \in \mathbb{R}$ is such that $v'(\xi) = 0$ and $0 < v(\xi) < 1$, then we distinguish between the following two alternatives:

Alt. 1. Suppose that $(0 <) s_0 \leq v(\xi) < 1$ and $v'(\xi) = 0$. By Hypothesis **(H1)**, the graph of the function

$$F(r) \stackrel{\text{def}}{=} \int_0^r f(s) \, ds \quad \text{for } r \in \mathbb{R}$$

shows that there is a unique number $s_1 \in (0, s_0)$ such that $F(s_1) = F(v(\xi))$, thanks to $F(s_0) \leq F(v(\xi)) < F(1) < 0 = F(0)$. From

$$F(v(\xi)) = V(\xi) \geq V(z) \geq F(v(z)) \quad \text{for all } z \geq \xi$$

and the strict monotonicity of the function F on the intervals $[s_1, s_0]$ (F strictly monotone decreasing) and $[s_0, 1]$ (F strictly monotone increasing) we deduce that

$$0 < s_1 \leq v(z) \leq v(\xi) < 1 \quad \text{for all } z \geq \xi.$$

But this is a contradiction to $\lim_{z \rightarrow +\infty} v(z) = 1$.

Alt. 2. Now suppose that $0 < v(\xi) < s_0 (< 1)$ and $v'(\xi) = 0$. Eq. (2.5) being equivalent with

$$(2.14) \quad e^{-cz} \cdot \frac{d}{dz} (e^{cz} v'(z)) = -f(v(z)), \quad z \in \mathbb{R},$$

we conclude that the function $z \mapsto e^{cz} v'(z) : \mathbb{R} \rightarrow \mathbb{R}$ is strictly monotone increasing on the maximal open interval $J \subset \mathbb{R}$ such that

$$\xi \in J \subset v^{-1}(0, s_0) \stackrel{\text{def}}{=} \{z \in \mathbb{R} : 0 < v(z) < s_0\}.$$

Consequently,

$$e^{cz} v'(z) < e^{c\xi} v'(\xi) = 0 \quad \text{holds for all } z \in J \cap (-\infty, \xi),$$

which gives $J \cap (-\infty, \xi] \cap (z_0, z_1) = \emptyset$ and $0 < v(\xi) < v(z) < s_0$ for all $z \in J \cap (-\infty, \xi)$. Similarly,

$$e^{cz} v'(z) > e^{c\xi} v'(\xi) = 0 \quad \text{for all } z \in J \cap (\xi, +\infty),$$

which gives $0 < v(\xi) < v(z) < s_0$ for all $z \in J \cap (\xi, +\infty)$. Hence, $\xi \in J$ is a strict (global) minimizer for the function v over the interval J . As a consequence, $0 < v(\xi) \leq v(z) < s_0 < 1$ holds for all $z \in J$. The limits in (2.6) force $-\infty < \inf J < \sup J < +\infty$, so that $J = (z'_0, z'_1)$ is a bounded open interval and $v(z'_0) = v(z'_1) = s_0$ together with $v'(z'_0) < v'(\xi) = 0 < v'(z'_1)$. Recalling $\lim_{z \rightarrow -\infty} v(z) = 0$, we now observe that v attains its maximum over the interval $(-\infty, z'_0)$ at some $\xi' \in (-\infty, z'_0)$, i.e., $0 \leq v(z) \leq v(\xi')$ for all $z \leq z'_0$, together with $s_0 < v(\xi') \leq 1$ and $v'(\xi') = 0$.

By Case (ii) above, the alternative $v(\xi') = 1$ would force $v(z) \equiv 1$ for all $z \geq \xi'$. We may set $z = z'_0 > \xi'$ to arrive at the following contradiction, $v(z'_0) = 1 > s_0 = v(z'_0)$. Consequently, we must have $s_0 < v(\xi') < 1$ and $v'(\xi') = 0$. But this situation for $\xi' \in \mathbb{R}$ corresponds to Alternative 1 for $\xi \in \mathbb{R}$ above which has already been excluded.

We combine Alternatives 1 and 2 to conclude that if $\xi \in \mathbb{R}$ is a number with $v'(\xi) = 0$ then we have either $v(\xi) = 0$ or else $v(\xi) = 1$. Furthermore, by Cases (i) and (ii) (treated before Alternatives 1 and 2), we have even $v(z) \equiv v(\xi) = 0$ for all $z \leq \xi$ (in Case (i)), or $v(z) \equiv v(\xi) = 1$ for all $z \geq \xi$ (in Case (ii)).

Now we are ready to complete our proof. It remains to treat the following two possibilities: (a) $z_0 > -\infty$ and $v'(z_0) = 0$, and (b) $z_1 < +\infty$ and $v'(z_1) = 0$.

We have $0 \leq v(z_0) < v(z) < v(z_1) \leq 1$ for all $z \in (z_0, z_1)$. We can have neither $0 < v(z_0) < 1$ nor $0 < v(z_1) < 1$, by Alternatives 1 and 2.

Concerning (a), we must have $v(z_0) = 0$ and, thus, $v(z) \equiv v(z_0) = 0$ for all $z \leq z_0$, as well, by Case (i).

Similarly, concerning (b), we must have $v(z_1) = 1$ and, thus, $v(z) \equiv v(z_1) = 1$ for all $z \geq z_1$, again, by Case (ii).

To conclude our proof, we recall that $v'(z) > 0$ holds for every $z \in (z_0, z_1)$. ■

Remark 2.2 Let $v : \mathbb{R} \rightarrow \mathbb{R}$ and $(z_0, z_1) \subset \mathbb{R}$ be as in Proposition 2.1 above. Since our problem (2.5), (2.6) is invariant with respect to a shift $z \mapsto z + \zeta$ in the variable $z \in \mathbb{R}$, by a fixed number $\zeta \in \mathbb{R}$, then also the function $\tilde{v} : \mathbb{R} \rightarrow \mathbb{R}$, $\tilde{v}(z) \stackrel{\text{def}}{=} v(z + \zeta)$ for $z \in \mathbb{R}$, is a C^2 -solution of eq. (2.5) satisfying $0 \leq \tilde{v} \leq 1$ and $\tilde{v}' \geq 0$ on \mathbb{R} , $\tilde{v}' > 0$ on $(\tilde{z}_0, \tilde{z}_1) = (z_0 - \zeta, z_1 - \zeta)$, and

$$(2.15) \quad \begin{cases} \lim_{z \rightarrow \tilde{z}_0+} \tilde{v}(z) = 0 & \text{and} & \tilde{v}(z) = 0 \text{ if } -\infty < z \leq \tilde{z}_0, \\ \lim_{z \rightarrow \tilde{z}_1-} \tilde{v}(z) = 1 & \text{and} & \tilde{v}(z) = 1 \text{ if } \tilde{z}_1 \leq z < +\infty. \end{cases}$$

We will see in the next paragraph (§2.2, Proposition 2.3) that problem (2.5), (2.6) has a C^2 -solution $v : \mathbb{R} \rightarrow \mathbb{R}$ for precisely one value of the speed $c \in \mathbb{R}$. Furthermore, this solution is monotone increasing and unique up to a shift by a constant $\zeta \in \mathbb{R}$, as described above.

2.2 Reduction to a First-Order O.D.E.

We use a phase plane transformation (cf. J. D. MURRAY [17], §13.2, pp. 440–441, and P. DRÁBEK and P. TAKÁČ [6]) in order to describe all monotone increasing travelling waves $u(x, t) = v(x - ct, t) \equiv U(x - ct - \zeta)$ where $U : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 -solution of problem (2.5), (2.6) as specified in Remark 2.2 and normalized by $U(0) = s_0$, and $\zeta \in \mathbb{R}$ is a suitable constant. Instead of trying to find such a unique function $U = U(z)$ of $z \in \mathbb{R}$, we will calculate the derivative dz/dU of its inverse function $U \mapsto z = z(U)$ as a function of $U \in (0, 1)$. In fact, below we find a nonlinear differential equation for the derivative

$$U'(z) = \left(\frac{dz}{dU} \right)^{-1} \equiv \frac{1}{z'(U)} \quad \text{as a function of } U \in (0, 1).$$

To this end, we make the substitution

$$(2.16) \quad V \stackrel{\text{def}}{=} \frac{dU}{dz} > 0 \quad \text{for } z \in (z_0, z_1)$$

and consequently look for $V = V(U)$ as a function of $U \in (0, 1)$ that satisfies the following differential equation obtained from eq. (2.4):

$$\frac{dV}{dU} \cdot \frac{dU}{dz} + c \frac{dU}{dz} + f(U) = 0, \quad z \in (z_0, z_1),$$

that is,

$$(2.17) \quad \frac{dV}{dU} \cdot V + cV + f(U) = 0, \quad U \in (0, 1).$$

Hence, we are looking for the inverse function $U \mapsto z(U)$ with the derivative

$$\frac{dz}{dU} = \frac{1}{V(U)} > 0 \quad \text{for } U \in (0, 1), \quad \text{such that } z(s_0) = 0.$$

Finally, in equation (2.17) we make the substitution

$$(2.18) \quad y = V^2 = \left| \frac{dU}{dz} \right|^2 > 0$$

and write r in place of U , thus arriving at

$$\frac{1}{2} \cdot \frac{dy}{dr} + c\sqrt{y} + f(r) = 0, \quad r \in (0, 1).$$

This means that the unknown function $y : (0, 1) \rightarrow (0, \infty)$ of r verifies also

$$(2.19) \quad \frac{dy}{dr} = -2 \left(c\sqrt{y} + f(r) \right), \quad r \in (0, 1),$$

where $y^+ = \max\{y, 0\}$. Since we require that $U = U(z)$ be sufficiently smooth, at least continuously differentiable, with $U'(z) \rightarrow 0$ as $z \rightarrow z_0+$ and $z \rightarrow z_1-$, the function $y = y(r) = |dU/dz|^2$ must satisfy the boundary conditions

$$(2.20) \quad y(0) = y(1) = 0.$$

We take advantage of Hypotheses **(H1)** on f to claim that, by a result in P. DRÁBEK and P. TAKÁČ [6, Corollary 5.5], the overdetermined first-order, two-point boundary value problem (2.19), (2.20) admits a C^1 -solution $y : [0, 1] \rightarrow \mathbb{R}$ for precisely one value of $c \in \mathbb{R}$; this solution is positive in $(0, 1)$. We emphasize again that the nonlinearity $y \mapsto \sqrt{y^+}$ in the differential equation (2.19) does not satisfy a local Lipschitz condition, so, due to the lack of uniqueness of a solution, the standard shooting method cannot be applied directly. Finally, the desired TW-solution $U = U(z)$ is calculated from eq. (2.16) with $V = \sqrt{y}$. More precisely, the function U , which is monotone increasing, is calculated by integrating the differential equation

$$(2.21) \quad dz = \frac{dU}{\sqrt{y(U)}} \quad \text{for } U \in (0, 1), \quad \text{with } z(s_0) = 0.$$

We summarize our results from paragraphs §2.1 and §2.2 in the following proposition:

Proposition 2.3 (TW-solutions.) *Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies Hypothesis **(H1)**. Then the following statements are valid:*

- (a) *The two-point boundary value problem (2.19), (2.20) has a C^1 -solution $y : [0, 1] \rightarrow \mathbb{R}$ for precisely one value of the speed $c \in \mathbb{R}$; this critical value, denoted by c^* , is positive, $c^* > 0$. The corresponding solution, $y \equiv y_{c^*}$, is unique and positive in $(0, 1)$.*
- (b) *Problem (2.5), (2.6) with $c \in \mathbb{R}$ has a C^2 -solution $v : \mathbb{R} \rightarrow \mathbb{R}$ if and only if $c = c^*$. This solution, $v \equiv v_{c^*}$, is unique if we require also $v(0) = s_0$; we denote it by $U \equiv v_{c^*}$.*
- (c) *The solution $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $0 \leq U(z) \leq 1$ and $U'(z) \geq 0$ for every $z \in \mathbb{R}$. Moreover, there is an open interval $(z_0, z_1) \subset \mathbb{R}$, $-\infty \leq z_0 < z_1 \leq +\infty$, such that $0 < U(z) < 1$ and $U'(z) > 0$ hold for every $z \in (z_0, z_1)$, and*

$$(2.22) \quad \begin{cases} \lim_{z \rightarrow z_0+} U(z) = 0 & \text{and} & U(z) = 0 \text{ if } -\infty < z \leq z_0, \\ \lim_{z \rightarrow z_1-} U(z) = 1 & \text{and} & U(z) = 1 \text{ if } z_1 \leq z < +\infty. \end{cases}$$

- (d) *Without the condition $v(0) = s_0$, all other solutions $v : \mathbb{R} \rightarrow \mathbb{R}$ of problem (2.5), (2.6) with $c = c^*$ are given by $v(z) = U(z + \zeta)$ for $z \in \mathbb{R}$, where $\zeta \in \mathbb{R}$ is arbitrary.*

2.3 The Asymptotic Shape of Travelling Waves

The asymptotic shape of the TW-solution $U(z)$ as $z \rightarrow \mp\infty$ is determined by the asymptotic behavior of the integral

$$(2.23) \quad \int_{r_0}^{r_1} \frac{dr}{V(r)} \quad \text{as } r_0 \rightarrow 0+ \text{ and } r_1 \rightarrow 1-, \text{ respectively;}$$

cf. eq. (2.21). Indeed, we observe that the integral

$$(2.24) \quad z(U) = z(s_0) + \int_{s_0}^U \frac{dr}{V(r)}, \quad \text{for } U \in (0, 1),$$

renders the inverse function of a TW-solution $U(z)$ that is determined uniquely by the point $z(s_0) = 0$ at which $U(0) = s_0$. Next, let us consider the limits

$$(2.25) \quad z_0 \stackrel{\text{def}}{=} \lim_{U \rightarrow 0+} z(U) \geq -\infty \quad \text{and} \quad z_1 \stackrel{\text{def}}{=} \lim_{U \rightarrow 1-} z(U) \leq +\infty.$$

These limits can be finite or infinite, $-\infty \leq z_0 < 0 < z_1 \leq +\infty$, depending on whether the integral in (2.23) is convergent or divergent as $r_0 \rightarrow 0+$ and $r_1 \rightarrow 1-$, respectively. This, in turn, depends on the asymptotic behavior of the function $f(r)$ as $r \rightarrow 0+$ and $r \rightarrow 1-$, thanks to $y = V^2$ being a solution to problem (2.19), (2.20).

Example 2.4 From a combination of (2.19), (2.20), (2.24), and (2.25) one can deduce that $z_0 > -\infty$ occurs if the reaction term $f(r)$ has the following asymptotic behavior as $r \rightarrow 0+$:

$$(2.26) \quad \lim_{r \rightarrow 0+} \frac{f(r)}{r^{\alpha_0}} = -\gamma_0 < 0 \quad \text{where} \quad 0 < \alpha_0 < 1.$$

Analogously, $z_1 < +\infty$ occurs if $f(r)$ has the following asymptotic behavior as $r \rightarrow 1-$:

$$(2.27) \quad \lim_{r \rightarrow 1-} \frac{f(r)}{(1-r)^{\alpha_1}} = \gamma_1 > 0 \quad \text{where} \quad 0 < \alpha_1 < 1.$$

These claims are proved in details in P. DRÁBEK and P. TAKÁČ [6], Corrolary 6.3, Part (i). Notice that such a function f cannot be differentiable or Lipschitzian at the point $r = 0$ or $r = 1$, respectively.

It is well-known that in the classical case of $f : \mathbb{R} \rightarrow \mathbb{R}$ being continuously differentiable, one has $z_0 = -\infty$ and $z_1 = +\infty$, see P. C. FIFE and J. B. MCLEOD [7, Sect. 1] and J. D. MURRAY [17], §13.3, pp. 444–449. At this point, we do not distinguish between the cases z_0 or z_1 being finite or infinite. Towards the end of this article, when investigating the ω -limit sets of solutions to the initial value problem (2.1), we will focus on the new case $-\infty < z_0 < z_1 < \infty$ only, which has not yet been treated in the literature.

3 Solutions of the Initial Value Problem (2.1)

In order to be able to establish the Hölder continuity of all partial derivatives that appear in eq. (1.1) above, we impose the following Hölder continuity hypothesis on the reaction function f :

(H2) $f : \mathbb{R} \rightarrow \mathbb{R}$ is an α -Hölder continuous function, with an exponent $\alpha \in (0, 1)$.

We investigate the existence and uniqueness of a classical solution $v(z, t)$ to the Cauchy (initial value) problem (2.1) obtained from problem (1.1). We seek bounded (L^∞ -) classical solutions “squeezed” (i.e., bounded below and above, respectively) between an ordered pair of a sub- and supersolution to the initial value problem (2.1). Such a solution takes values $v(z, t) \in [0, 1]$ for all $z \in \mathbb{R}$ and for all $t \in (0, \infty)$. The initial condition is satisfied in the sense of the weak*-limit in $L^\infty(\mathbb{R})$. In order to obtain the desired regularity properties of a *bounded classical solution* to the initial value problem (2.1), below we use the Green function associated with the linear part of this problem.

3.1 Mild and Classical Solutions

First, we denote by

$$(3.1) \quad G(x, y; t) \equiv G(|x - y|; t) \stackrel{\text{def}}{=} \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{|x - y|^2}{4t}\right) \\ \text{for } x, y \in \mathbb{R} \text{ and } t \in (0, \infty)$$

the Green function for the standard heat equation (cf. (1.1)). The desired Green function for the linear analogue of problem (2.1),

$$(3.2) \quad \begin{cases} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial z^2} - c \frac{\partial v}{\partial z} = g(z, t), & (z, t) \in \mathbb{R} \times \mathbb{R}_+; \\ v(z, 0) = v_0(z), \end{cases}$$

where we have replaced the nonlinearity $f(v(z, t))$ by a given function $g \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, is obtained by shifting the space variable $z \mapsto x = z + ct$ in eq. (3.1),

$$(3.3) \quad G^{(c)}(z, y; t) \equiv G^{(c)}(z - y; t) \stackrel{\text{def}}{=} G(|z - y + ct|; t) \\ = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{|z - y + ct|^2}{4t}\right) \quad \text{for } z, y \in \mathbb{R} \text{ and } t \in (0, \infty).$$

Second, we require that any bounded classical solution $v : \mathbb{R} \times \mathbb{R}_+ \rightarrow [0, 1] \subset \mathbb{R}$ to problem (2.1) be also a *mild L^∞ -solution*, i.e., v must be essentially bounded and obey the following integral equation:

$$(3.4) \quad v(z, t) = [\mathcal{G}_1(t)v_0](z) + [\mathcal{G}_2(f \circ v)](z, t) \quad \text{for } (z, t) \in \mathbb{R} \times (0, \infty),$$

where $\mathcal{G}_1(t)$ and \mathcal{G}_2 are integral operators defined by

$$(3.5) \quad [\mathcal{G}_1(t)g_0](z) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} G^{(c)}(z, y; t) g_0(y) dy \quad \text{and}$$

$$(3.6) \quad [\mathcal{G}_2g](z, t) \stackrel{\text{def}}{=} \int_0^t \int_{-\infty}^{\infty} G^{(c)}(z, y; t - s) g(y, s) dy ds$$

for $(z, t) \in \mathbb{R} \times (0, \infty)$ and for all functions $g_0 \in L^\infty(\mathbb{R})$ and $g \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$.

It is easy to see that the first operator, $\mathcal{G}_1(t)$, has the following boundedness property: Given any nonnegative integers $k, m \in \mathbb{Z}_+$, there exists a constant $M_{k,m} \in \mathbb{R}_+$ such that

$$(3.7) \quad \left| \frac{\partial^{k+m}}{\partial t^k \partial z^m} [\mathcal{G}_1(t)g_0](z) \right| \leq M_{k,m} t^{-k-(m/2)} \cdot \|g_0\|_{L^\infty(\mathbb{R})}$$

holds for all $(z, t) \in \mathbb{R} \times (0, \infty)$.

This estimate follows directly from O. A. LADYZHENSKAYA, N. N. URAL'TSEVA, and V. A. SOLONNIKOV [13, Sect. IV, §1], ineq. (2.5) on p. 274; the partial derivative on the left-hand side is taken pointwise in the classical sense. The second operator, \mathcal{G}_2 , is a bit more complicated: Given any positive number $T \in (0, \infty)$, there exists a constant $M^{(T)} \in \mathbb{R}_+$ such that

$$(3.8) \quad \left| \frac{\partial}{\partial z} [\mathcal{G}_2 g](z, t) \right| \leq M^{(T)} \|g\|_{L^\infty(\mathbb{R} \times (0, T))} \quad \text{and}$$

$$(3.9) \quad \frac{|[\mathcal{G}_2 g](z, t') - [\mathcal{G}_2 g](z, t)|}{|t' - t|^{1/2}} \leq M^{(T)} \|g\|_{L^\infty(\mathbb{R} \times (0, T))}$$

hold for all $(z, t), (z, t') \in \mathbb{R} \times [0, T]$, $t \neq t'$.

Also these estimates follow directly from [13, Sect. IV, §1], p. 263.

Finally, we quote the following regularity result for $\mathcal{G}_2 g$ proved in [13, Sect. IV, §2], ineq. (2.1) on p. 273: Let $\ell \in (0, 1)$ be arbitrary. If $g \in C^{\ell, \ell/2}(\mathbb{R} \times [0, T])$ then all partial derivatives

$$\frac{\partial}{\partial z} [\mathcal{G}_2 g], \quad \frac{\partial^2}{\partial z^2} [\mathcal{G}_2 g] \quad \text{and} \quad \frac{\partial}{\partial t} [\mathcal{G}_2 g]$$

belong to the Hölder space $C^{\ell, \ell/2}(\mathbb{R} \times [0, T])$ defined below. Furthermore, there is a constant $M_\ell^{(T)} \in \mathbb{R}_+$, independent from $g \in C^{\ell, \ell/2}(\mathbb{R} \times [0, T])$, such that

$$(3.10) \quad \max \left\{ \left\| \frac{\partial}{\partial z} [\mathcal{G}_2 g] \right\|_{C^{\ell, \ell/2}(\mathbb{R} \times [0, T])}, \left\| \frac{\partial^2}{\partial z^2} [\mathcal{G}_2 g] \right\|_{C^{\ell, \ell/2}(\mathbb{R} \times [0, T])}, \left\| \frac{\partial}{\partial t} [\mathcal{G}_2 g] \right\|_{C^{\ell, \ell/2}(\mathbb{R} \times [0, T])} \right\} \leq M_\ell^{(T)} \|g\|_{C^{\ell, \ell/2}(\mathbb{R} \times [0, T])}.$$

This Hölder space is defined as follows. Given any number $\ell \in (0, 1)$, we denote by $C^{\ell, \ell/2}(\mathbb{R} \times [0, T])$ the Banach space of all bounded functions $g : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ such that

$$(3.11) \quad [g]^{(\ell, \ell/2)} \stackrel{\text{def}}{=} \sup_{0 < |z' - z| \leq 1, t \in [0, T]} \frac{|g(z', t) - g(z, t)|}{|z' - z|^\ell} + \sup_{z \in \mathbb{R}, 0 \leq t < t' \leq T} \frac{|g(z, t') - g(z, t)|}{|t' - t|^{\ell/2}} < \infty.$$

The norm in $C^{\ell, \ell/2}(\mathbb{R} \times [0, T])$ is defined by

$$(3.12) \quad \|g\|_{C^{\ell, \ell/2}(\mathbb{R} \times [0, T])} \stackrel{\text{def}}{=} [g]^{(\ell, \ell/2)} + \|g\|_{L^\infty(\mathbb{R} \times [0, T])}.$$

These regularity results for the integral operators $\mathcal{G}_1(t)$ and \mathcal{G}_2 yield the following differentiability properties for any mild L^∞ -solution v of the integral equation (3.4): Let $0 < t_0 < \tau < T < \infty$ and let us begin with an arbitrary function $g \in L^\infty(\mathbb{R} \times (0, T))$ in place of $f \circ v$. Then the function $v(z, t)$ on the left-hand side of eq. (3.4) satisfies

$$(3.13) \quad \left| \frac{\partial v}{\partial z}(z, t) \right| \leq C_{t_0}^{(T)} (\|v_0\|_{L^\infty(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R} \times (0, T))}) \quad \text{and}$$

$$(3.14) \quad \frac{|v(z, t') - v(z, t)|}{|t' - t|^{1/2}} \leq C_{t_0}^{(T)} (\|v_0\|_{L^\infty(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R} \times (0, T))})$$

for all $(z, t), (z, t') \in \mathbb{R} \times [t_0, T], t \neq t',$

where $C_{t_0}^{(T)} \in \mathbb{R}_+$ is a constant independent from $v_0 \in L^\infty(\mathbb{R})$ and $g \in L^\infty(\mathbb{R} \times (0, T))$. These estimates follow from an easy combination of the inequalities in (3.7), (3.8), and (3.9), with the interval $[0, T]$ replaced by $[t_0, T]$.

The estimates below take advantage of the fact that the linear initial value problem (3.2), with any given inhomogeneity $g \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ in place of $f \circ v$ on the right-hand side, possesses a unique mild L^∞ -solution $v : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$. In particular, this solution,

$$v(z, t) = [\mathcal{G}_1(t)v_0](z) + [\mathcal{G}_2g](z, t), \quad \text{for } (z, t) \in \mathbb{R} \times (0, \infty),$$

satisfies (3.13) and (3.14). Hence, recalling our Hypothesis **(H2)** (i.e., $f : \mathbb{R} \rightarrow \mathbb{R}$ is α -Hölder-continuous) and setting $\tilde{g} = f \circ v$, (3.13) and (3.14) imply $\tilde{g} \in C^{\alpha, \alpha/2}(\mathbb{R} \times [t_0, T])$. We denote by \tilde{v} the unique mild L^∞ -solution of problem (3.2) in the domain $\mathbb{R} \times (t_0, T)$,

$$\begin{cases} \frac{\partial \tilde{v}}{\partial t} - \frac{\partial^2 \tilde{v}}{\partial z^2} - c \frac{\partial \tilde{v}}{\partial z} = \tilde{g}, & (z, t) \in \mathbb{R} \times (t_0, T); \\ \tilde{v}(z, t_0) = v(z, t_0). \end{cases}$$

Recall that the initial value $v(\cdot, t_0) \in L^\infty(\mathbb{R})$ satisfies (3.13). Here, we have replaced the initial time $t = 0$ by $t = t_0 \in (0, \tau)$. We combine the regularity estimates in (3.7) and (3.10), thus arriving at

$$(3.15) \quad \begin{aligned} & \max \left\{ \left\| \frac{\partial \tilde{v}}{\partial z} \right\|_{C^{\alpha, \alpha/2}(\mathbb{R} \times [\tau, T])}, \left\| \frac{\partial^2 \tilde{v}}{\partial z^2} \right\|_{C^{\alpha, \alpha/2}(\mathbb{R} \times [\tau, T])}, \left\| \frac{\partial \tilde{v}}{\partial t} \right\|_{C^{\alpha, \alpha/2}(\mathbb{R} \times [\tau, T])} \right\} \\ & \leq C_\alpha^{(t_0, \tau, T)} (\|v(\cdot, t_0)\|_{L^\infty(\mathbb{R})} + \|\tilde{g}\|_{C^{\alpha, \alpha/2}(\mathbb{R} \times [t_0, T])}) \\ & \leq C_\alpha^{(\tau, T)} (\|v\|_{L^\infty(\mathbb{R} \times (0, T))} + \|g\|_{L^\infty(\mathbb{R} \times (0, T))}) , \end{aligned}$$

where $C_\alpha^{(t_0, \tau, T)} \in \mathbb{R}_+$ is a constant independent from $v(\cdot, t_0) \in L^\infty(\mathbb{R})$ and $\tilde{g} \in C^{\alpha, \alpha/2}(\mathbb{R} \times [t_0, T])$, and $C_\alpha^{(\tau, T)} \in \mathbb{R}_+$ is another constant independent from $v \in L^\infty(\mathbb{R} \times (0, T))$ and $g \in L^\infty(\mathbb{R} \times (0, T))$. To derive the second inequality in (3.15), we have made use of (3.13) and (3.14).

We conclude from (3.15) that any mild L^∞ -solution v of problem (2.1) satisfies

$$\frac{\partial v}{\partial z}, \frac{\partial^2 v}{\partial z^2}, \frac{\partial v}{\partial t} \in C^{\alpha, \alpha/2}(\mathbb{R} \times [\tau, T]) \quad \text{whenever } 0 < \tau < T < \infty,$$

and

$$(3.16) \quad \max \left\{ \left\| \frac{\partial v}{\partial z} \right\|_{C^{\alpha, \alpha/2}(\mathbb{R} \times [\tau, T])}, \left\| \frac{\partial^2 v}{\partial z^2} \right\|_{C^{\alpha, \alpha/2}(\mathbb{R} \times [\tau, T])}, \left\| \frac{\partial v}{\partial t} \right\|_{C^{\alpha, \alpha/2}(\mathbb{R} \times [\tau, T])} \right\} \\ \leq C_{\alpha}^{(\tau, T)} (\|v\|_{L^{\infty}(\mathbb{R} \times (0, T))} + \|f \circ v\|_{L^{\infty}(\mathbb{R} \times (0, T))}) .$$

Consequently, v is also a *bounded classical solution* to problem (2.1); the initial condition is satisfied in the sense of the weak*-limit $v(\cdot, t) \xrightarrow{*} v_0$ in $L^{\infty}(\mathbb{R})$ as $t \rightarrow 0+$.

3.2 Weak Sub- and Supersolutions and Weak Solutions

Let $v_0 \in L^{\infty}(\mathbb{R})$. A function $\underline{v} \in L^{\infty}(\mathbb{R} \times \mathbb{R}_+)$ will be called a *weak L^{∞} -subsolution* of the initial value problem (2.1) if it satisfies the following three conditions:

- (i) For every nonnegative test function $\phi \in W_0^{1,1}(\mathbb{R})$, we have

$$\limsup_{t \rightarrow 0+} \int_{-\infty}^{\infty} [\underline{v}(z, t) - v_0(z)] \phi(z) dz \leq 0 .$$

- (ii) \underline{v} is Lipschitz-continuous in every set $\mathbb{R} \times [\tau, T]$ whenever $0 < \tau < T < \infty$, i.e., $\partial \underline{v} / \partial z, \partial \underline{v} / \partial t \in L^{\infty}(\mathbb{R} \times [\tau, T])$.

- (iii) For every nonnegative test function $\phi \in W_0^{1,1}(\mathbb{R})$, the following inequality holds for a.e. $t \in (0, \infty)$,

$$(3.17) \quad \frac{d}{dt} \int_{-\infty}^{\infty} \underline{v}(z, t) \phi(z) dz + \int_{-\infty}^{\infty} \frac{\partial \underline{v}}{\partial z}(z, t) \cdot \frac{\partial \phi}{\partial z}(z) dz \\ - c \int_{-\infty}^{\infty} \frac{\partial \underline{v}}{\partial z}(z, t) \cdot \phi(z) dz \leq \int_{-\infty}^{\infty} f(\underline{v}(z, t)) \phi(z) dz .$$

A *weak L^{∞} -supersolution* $\bar{v} \in L^{\infty}(\mathbb{R} \times \mathbb{R}_+)$ is defined analogously. We say that a function $v \in L^{\infty}(\mathbb{R} \times \mathbb{R}_+)$ is a *weak L^{∞} -solution* of the initial value problem (2.1) if and only if v is a weak L^{∞} -sub- and -supersolution to (2.1). Of course, in Condition (i), the nonnegative test functions $\phi \in W_0^{1,1}(\mathbb{R})$ may be replaced by nonnegative functions $\phi \in L^1(\mathbb{R})$. Condition (ii) combined with equality in (3.17) implies $\partial v / \partial z, \partial v / \partial t, \partial^2 v / \partial z^2 \in L^{\infty}(\mathbb{R} \times [\tau, T])$.

We remark that any weak L^{∞} -solution of problem (2.1) is also a mild L^{∞} -solution and vice versa; cf. J. M. BALL [2, Theorem, p. 371] or A. PAZY [18, Theorem, p. 259]. Thanks to the regularity properties in (3.16), a mild L^{∞} -solution is also a bounded classical solution. It is shown in A. PAZY [18, §4.2, pp. 105–110] that any bounded classical solution is also a mild L^{∞} -solution.

All types of sub- and supersolutions and solutions of problem (2.1), defined above for all times $t \in \mathbb{R}_+$, can be defined analogously for time $t \in [\tau, T)$ from a bounded time interval $[\tau, T) \subset \mathbb{R}_+$.

The following *one-sided Lipschitz condition* is a crucial hypothesis imposed on the function f in our method for establishing the *weak comparison principle* for weak L^∞ -sub- and -supersolutions:

(H3) There is a number $L \in \mathbb{R}_+$ such that

$$(3.18) \quad f(s') - f(s) \leq L(s' - s) \quad \text{for all } s, s' \in \mathbb{R}, \quad s < s'.$$

Lemma 3.1 (Weak comparison principle.) *Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfy Hypothesis **(H3)**. Assume that $\underline{v}, \bar{v} \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a pair of weak L^∞ -sub- and -supersolutions to problem (2.1), such that $\underline{v}(\cdot, 0) = \underline{v}_0 \leq \bar{v}_0 = \bar{v}(\cdot, 0)$ in $L^\infty(\mathbb{R})$. Then we have also $\underline{v} \leq \bar{v}$ a.e. in $\mathbb{R} \times \mathbb{R}_+$.*

Proof. We subtract the analogue of ineq. (3.17) for a supersolution (with the reversed inequality) from (3.17) for a subsolution,

$$(3.19) \quad \begin{aligned} & \frac{d}{dt} \int_{-\infty}^{\infty} (\underline{v}(z, t) - \bar{v}(z, t)) \phi(z) dz + \int_{-\infty}^{\infty} \frac{\partial}{\partial z} (\underline{v} - \bar{v}) \cdot \frac{\partial \phi}{\partial z}(z) dz \\ & - c \int_{-\infty}^{\infty} \frac{\partial}{\partial z} (\underline{v} - \bar{v}) \cdot \phi(z) dz \\ & \leq \int_{-\infty}^{\infty} [f(U(z) + \underline{v}(z)) - f(U(z) + \bar{v}(z))] \phi(z) dz, \end{aligned}$$

for every nonnegative test function $\phi \in W_0^{1,1}(\mathbb{R})$. Recalling Conditions (i) and (ii), we observe that

$$\frac{d}{dt} \int_{-\infty}^{\infty} (\underline{v}(z, t) - \bar{v}(z, t)) \phi(z) dz = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\underline{v} - \bar{v}) \cdot \phi(z) dz$$

holds for a.e. $t \in (0, \infty)$.

Let $(\underline{v} - \bar{v})^+ = \max\{(\underline{v} - \bar{v}), 0\}$ denote the nonnegative part of the difference $\underline{v} - \bar{v}$. By standard arguments, we have $(\underline{v} - \bar{v})^+ \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ and

$$\frac{\partial}{\partial t} (\underline{v} - \bar{v})^+, \quad \frac{\partial}{\partial z} (\underline{v} - \bar{v})^+ \in L^\infty(\mathbb{R} \times [\tau, T]) \quad \text{whenever } 0 < \tau < T < \infty.$$

For a.e. $t \in (0, \infty)$, in ineq. (3.19) above we may replace ϕ by the product $(\underline{v} - \bar{v})^+(\cdot, t) \phi$, thus obtaining

$$\begin{aligned} & \frac{1}{2} \cdot \frac{d}{dt} \int_{-\infty}^{\infty} [(\underline{v} - \bar{v})^+(z, t)]^2 \cdot \phi(z) dz \\ & + \int_{-\infty}^{\infty} \frac{\partial}{\partial z} (\underline{v} - \bar{v}) \cdot \frac{\partial}{\partial z} [(\underline{v} - \bar{v})^+(z, t) \phi(z)] dz \\ & - c \int_{-\infty}^{\infty} \frac{\partial}{\partial z} (\underline{v} - \bar{v}) \cdot (\underline{v} - \bar{v})^+(z, t) \phi(z) dz \\ & \leq \int_{-\infty}^{\infty} [f(U(z) + \underline{v}(z)) - f(U(z) + \bar{v}(z))] (\underline{v} - \bar{v})^+(z, t) \phi(z) dz. \end{aligned}$$

We simplify the integrands and apply inequality (3.18) from the one-sided Lipschitz condition in Hypothesis **(H3)** to the last integral, thus arriving at

$$\begin{aligned}
(3.20) \quad & \frac{1}{2} \cdot \frac{d}{dt} \int_{-\infty}^{\infty} [(\underline{v} - \bar{v})^+(z, t)]^2 \cdot \phi(z) \, dz \\
& + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial}{\partial z} [(\underline{v} - \bar{v})^+(z, t)]^2 \cdot \frac{\partial \phi}{\partial z}(z) \, dz + \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial z} (\underline{v} - \bar{v})^+ \right]^2 \cdot \phi(z) \, dz \\
& - \frac{c}{2} \int_{-\infty}^{\infty} \frac{\partial}{\partial z} [(\underline{v} - \bar{v})^+(z, t)]^2 \cdot \phi(z) \, dz \\
& \leq L \int_{-\infty}^{\infty} [(\underline{v} - \bar{v})^+(z, t)]^2 \cdot \phi(z) \, dz.
\end{aligned}$$

We conclude that the nonnegative function $W(z, t) = e^{2Lt} [(\underline{v} - \bar{v})^+(z, t)]^2$ satisfies the inequality

$$(3.21) \quad \begin{cases} \frac{\partial W}{\partial t} - \frac{\partial^2 W}{\partial z^2} - c \frac{\partial W}{\partial z} \leq 0, & (z, t) \in \mathbb{R} \times \mathbb{R}_+; \\ W(z, 0) = 0, \end{cases}$$

in the weak sense, $W \in L^\infty(\mathbb{R} \times (0, T))$ for every $T \in (0, \infty)$. Consequently, we substitute $x = z + ct$ and apply the weak maximum principle for the heat equation to the function $W(z + ct, t)$, which yields $W \leq 0$ a.e. in $\mathbb{R} \times \mathbb{R}_+$; hence, $(\underline{v} - \bar{v})^+ \equiv 0$ a.e. in $\mathbb{R} \times \mathbb{R}_+$.

The lemma is proved. ■

Lemma 3.1 has the following straight-forward consequence.

Corollary 3.2 (Uniqueness.) *Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfy Hypothesis **(H3)**. Then the initial value problem (2.1) has at most one weak L^∞ -solution. In particular, the same uniqueness result holds also for a bounded classical solution, as well. Finally, if also Hypothesis **(H2)** is satisfied, then this uniqueness result applies also to any mild L^∞ -solution.*

We give the proof of the existence of a weak L^∞ -solution to the initial value problem (2.1) in the next paragraph (§3.3).

3.3 Existence of a Solution to Problem (2.1)

The following existence result complements our uniqueness result from Corollary 3.2.

Proposition 3.3 *Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy Hypotheses **(H2)** and **(H3)**. Assume that the initial data $v_0 : \mathbb{R} \rightarrow \mathbb{R}$ are Lebesgue-measurable and satisfy $0 \leq v_0 \leq 1$*

a.e. on \mathbb{R} . Then the initial value problem (2.1) possesses a unique mild L^∞ -solution, say, $v : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$. In particular, the same existence and uniqueness result holds also for weak L^∞ -solutions and bounded classical solutions, as well. Finally, $0 \leq v(z, t) \leq 1$ holds for all $(z, t) \in \mathbb{R} \times (0, \infty)$.

Proof. We apply the well-known Tikhonov fixed point theorem (K. DEIMLING [3, Theorem 10.1, p. 90]) as follows. Given any $T \in (0, \infty)$, we denote by

$$\mathcal{X} = C_b(\mathbb{R} \times [0, T]) \stackrel{\text{def}}{=} L^\infty(\mathbb{R} \times (0, T)) \cap C(\mathbb{R} \times [0, T])$$

the vector space of all bounded continuous functions $v : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ endowed with the locally convex topology of uniform convergence on every compact set $K = J \times [0, T] \subset \mathbb{R} \times [0, T]$, where $J = [a, b] \subset \mathbb{R}$ is a compact interval. Thus, \mathcal{X} is a Fréchet space.

By our definition of a mild L^∞ -solution $v : \mathbb{R} \times \mathbb{R}_+ \rightarrow [0, 1] \subset \mathbb{R}$ to problem (2.1), v must be essentially bounded and obey the integral equation (3.4). Accordingly, we split it as

$$(3.22) \quad v(z, t) = [\mathcal{G}_1(t)v_0](z) + w(z, t) \quad \text{for } (z, t) \in \mathbb{R} \times [0, T],$$

where $w \in \mathcal{X}$ is a fixed point of the self-mapping $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ defined by

$$(3.23) \quad (\mathcal{T}w)(z, t) \stackrel{\text{def}}{=} [\mathcal{G}_2(f \circ v)](z, t) \quad \text{for } (z, t) \in \mathbb{R} \times [0, T],$$

with v defined in eq. (3.22), $v \in L^\infty(\mathbb{R} \times (0, T))$.

We recall that $\mathcal{G}_1(t)$ and \mathcal{G}_2 are integral operators defined by eqs. (3.5) and (3.6), respectively. Next, let us consider the closed convex subset

$$\mathcal{C} = \{w \in \mathcal{X} : |w(z, t)| \leq 1 \text{ for all } (z, t) \in \mathbb{R} \times [0, T]\}$$

of \mathcal{X} and denote

$$M = \max_{-1 \leq s \leq 2} |f(s)|; \quad 0 < M < \infty.$$

Recall that $0 \leq v_0 \leq 1$ on \mathbb{R} and the kernel of the integral operators $\mathcal{G}_1(t)$ and \mathcal{G}_2 satisfies $G^{(c)}(z, y; t) > 0$ and $\int_{-\infty}^{\infty} G^{(c)}(z, y; t) dy = 1$. Consequently, for every $w \in \mathcal{C}$ we have

$$0 \leq [\mathcal{G}_1(t)v_0](z) \leq 1, \quad -1 \leq v(z, t) = [\mathcal{G}_1(t)v_0](z) + w(z, t) \leq 2,$$

and $|f(v(z, t))| \leq M \quad \text{for } (z, t) \in \mathbb{R} \times [0, T].$

Applying these inequalities to eq. (3.23), we arrive at

$$|(\mathcal{T}w)(z, t)| = |[\mathcal{G}_2(f \circ v)](z, t)| \leq Mt \quad \text{for } (z, t) \in \mathbb{R} \times [0, T]$$

and for every function $w \in \mathcal{C}$. We take $T = 1/M \in (0, \infty)$ and observe that \mathcal{T} maps \mathcal{C} into itself.

From the uniform continuity of f on the compact interval $[-1, 2]$ and the properties of the kernel $G^{(c)}(z, y; t - s)$ of the integral operator \mathcal{G}_2 we deduce that $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ is continuous. Finally, we combine the regularity estimates (3.8) and (3.9) with Arzelà-Ascoli's compactness criterion in \mathcal{X} (P. DRÁBEK and J. MILOTA [5, Theorem 1.2.13, p. 32]) to conclude that the image $\mathcal{T}(\mathcal{C})$ is relatively compact. By Tikhonov's fixed point theorem, \mathcal{T} has a fixed point in \mathcal{C} , say, $\hat{w} \in \mathcal{C}$. The sum corresponding to eq. (3.22),

$$\begin{aligned}\hat{v}(z, t) &= [\mathcal{G}_1(t)v_0](z) + \hat{w}(z, t) \\ &= [\mathcal{G}_1(t)v_0](z) + [\mathcal{G}_2(f \circ \hat{v})](z, t) \quad \text{for } (z, t) \in \mathbb{R} \times [0, T],\end{aligned}$$

is a mild L^∞ -solution to problem (2.1) on the bounded time interval $[0, T]$ in place of \mathbb{R}_+ and, by regularity results in §3.1, also a weak L^∞ -solution and a bounded classical solution. By an analogue of Lemma 3.1 (weak comparison principle) for the bounded time interval $[0, T]$, we have $0 \leq \hat{v}(z, t) \leq 1$ for all $(z, t) \in \mathbb{R} \times [0, T]$. As in the case of the weak comparison principle, the uniqueness of \hat{v} follows from an analogue of Corollary 3.2.

Repeating this procedure in every time interval $[t_0, t_0 + T]$ of length T , for each $t_0 \in \mathbb{R}_+$, we can construct a mild L^∞ -solution to problem (2.1) for all times $t \in \mathbb{R}_+$. This is also a weak L^∞ -solution and a bounded classical solution. Finally, the uniqueness follows from Corollary 3.2. ■

3.4 Sub- and Supersolutions Resulting from Travelling Waves

Of course, the constant functions $\underline{v} \equiv 0$ and $\bar{v} \equiv 1$ form a trivial ordered pair of weak L^∞ -sub- and -supersolutions to problem (2.1). Now we are ready to modify the TW-solutions in order to construct suitable nontrivial weak L^∞ -sub- and -supersolutions to problem (2.1), as defined in §3.2.

In addition to $f : \mathbb{R} \rightarrow \mathbb{R}$ being continuous, we assume that f satisfies also the following two “secant” conditions on the interval $[0, 1]$:

(H4) There exists a number $\eta_0 \in \mathbb{R}$,

$$(3.24) \quad 0 < \eta_0 < \frac{1}{3} \cdot \min\{s_0, 1 - s_0\},$$

with the following property: Given any $\eta \in (0, \eta_0]$, there are constants $\delta \in (0, \eta)$ and $\underline{\mu} > 0$, $\bar{\mu} > 0$, depending on η , such that the following two conditions hold:

$$(3.25) \quad \inf_{0 \leq s \leq \delta} [f(s) - f(s + q)] \geq \underline{\mu} q \quad \text{for all } q \in (0, s_0 - 2\eta],$$

$$(3.26) \quad \inf_{1 - \delta \leq s \leq 1} [f(s - q) - f(s)] \geq \bar{\mu} q \quad \text{for all } q \in (0, 1 - s_0 - 2\eta].$$

In particular, a continuously differentiable function f (the classical case [7], [17, §13.3]) satisfies (H4) whenever $f'(0) < 0$, $f'(s_0) > 0$, and $f'(1) < 0$. However, the secant conditions are satisfied under somewhat different hypotheses on the differentiability of f :

Example 3.4 Our Hypothesis **(H4)** above is satisfied if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $f(0) = f(s_0) = f(1) = 0$ for some $0 < s_0 < 1$, together with $f(s) < 0$ for every $s \in (0, s_0)$, $f(s) > 0$ for every $s \in (s_0, 1)$, and f is differentiable in $(0, \eta^*) \cup \{s_0\} \cup (1 - \eta^*, 1)$ for some $\eta^* \in \mathbb{R}$,

$$0 < \eta^* < \frac{1}{3} \cdot \min\{s_0, 1 - s_0\},$$

with the derivatives satisfying $f'(s_0) > 0$ and

$$(3.27) \quad \lim_{s \rightarrow 0+} f'(s) = \lim_{s \rightarrow 1-} f'(s) = -\infty.$$

The last two conditions guarantee inequalities (3.25) and (3.26), respectively, for $q > 0$ small enough, whereas condition $f'(s_0) > 0$ guarantees them for $q \in (0, s_0 - 2\eta]$ near $s_0 - 2\eta$ and for $q \in (0, 1 - s_0 - 2\eta]$ near $1 - s_0 - 2\eta$, respectively, if $\eta > 0$ is small enough, say, $0 < \eta \leq \eta_0 \leq \eta^*$. If $\eta_0 \leq q \leq s_0 - 2\eta_0$ then ineq. (3.25) follows from a combination of conditions $\lim_{s \rightarrow 0+} f'(s) = -\infty$ and $f'(s_0) > 0$ with $f(s) < 0$ for every $s \in (0, s_0)$. In this case $s + q$ satisfies $\eta_0 \leq s + q \leq s_0 - 2\eta_0 + \delta < s_0 - \eta_0$, thanks to $0 \leq s \leq \delta < \eta \leq \eta_0$. On the other hand, if $\eta_0 \leq q \leq 1 - s_0 - 2\eta_0$ then ineq. (3.26) follows from a combination of conditions $\lim_{s \rightarrow 1-} f'(s) = -\infty$ and $f'(s_0) > 0$ with $f(s) > 0$ for every $s \in (s_0, 1)$. In this case $s - q$ satisfies $s_0 + \eta_0 < s - q \leq 1 - \eta_0$, by

$$\begin{aligned} s_0 + \eta_0 &= (1 - \eta_0) - (1 - s_0 - 2\eta_0) \leq (1 - \eta) - (1 - s_0 - 2\eta_0) \\ &< (1 - \delta) - (1 - s_0 - 2\eta_0) \leq s - q \leq 1 - \eta_0, \end{aligned}$$

thanks to $1 - \eta_0 \leq 1 - \eta < 1 - \delta \leq s \leq 1$.

Finally, we assume:

(H5) The initial condition $v_0 \in L^\infty(\mathbb{R})$ in the Cauchy problem (2.1) is defined at every point $z \in \mathbb{R}$ and satisfies $0 \leq v_0(z) \leq 1$ together with

$$(3.28) \quad v_0(-\infty) \stackrel{\text{def}}{=} \limsup_{z \rightarrow -\infty} v_0(z) < s_0 < \liminf_{z \rightarrow +\infty} v_0(z) \stackrel{\text{def}}{=} v_0(+\infty).$$

Accordingly, the constant $\eta \in (0, \eta_0]$ in Hypothesis **(H4)** is chosen to be small enough, such that also

$$(3.29) \quad v_0(-\infty) \stackrel{\text{def}}{=} \limsup_{z \rightarrow -\infty} v_0(z) < s_0 - 2\eta < s_0 + 2\eta < \liminf_{z \rightarrow +\infty} v_0(z) \stackrel{\text{def}}{=} v_0(+\infty).$$

As in Theorem 1.1, we mean the monotone limits

$$v_0(-\infty) = \lim_{n \rightarrow \infty} \operatorname{ess\,sup}_{-\infty < z \leq -n} v_0(z) \quad \text{and} \quad v_0(+\infty) = \lim_{n \rightarrow \infty} \operatorname{ess\,inf}_{n \leq z < +\infty} v_0(z).$$

Following the proof of Lemma 4.1 in P. C. FIFE and J. B. MCLEOD [7, pp. 347–348], we construct an ordered pair of weak L^∞ -sub- and -supersolutions to eq. (2.1), $v_1(z, t) \leq v_2(z, t)$ and $v_1(z, 0) \leq v_0(z) \leq v_2(z, 0)$, having the special forms rendered by the TW-solution U . Recall that $0 < U < 1$ and $U' > 0$ in $(z_0, z_1) \subset \mathbb{R}$ together with $\lim_{z \rightarrow z_0+} U(z) = 0$ and $\lim_{z \rightarrow z_1-} U(z) = 1$.

Proposition 3.5 *Assume that f satisfies Hypotheses (H1) and (H4), and v_0 satisfies (H5). Let $U : \mathbb{R} \rightarrow \mathbb{R}$ denote the stationary solution of eq. (2.1) described in Proposition 2.3, Part (c). Then there exist some constants $\mu, \nu, q_{0,i} \in (0, \infty)$ and $\xi_{\infty,i} \in \mathbb{R}$ such that the functions*

$$(3.30) \quad v_1(z, t) = \max\{U(z - \xi_1(t)) - q_1(t), 0\} \quad \text{and}$$

$$(3.31) \quad v_2(z, t) = \min\{U(z + \xi_2(t)) + q_2(t), 1\} \quad \text{for } (z, t) \in \mathbb{R} \times \mathbb{R}_+,$$

where

$$(3.32) \quad q_i(t) = q_{0,i} e^{-\mu t} \quad \text{and} \quad \xi_i(t) = \xi_{\infty,i} - \nu q_i(t) \quad \text{for } t \geq 0; \quad i = 1, 2,$$

are weak L^∞ -sub- and -supersolutions of the Cauchy problem (2.1), respectively. Finally, we have $0 \leq v_1(z, t) \leq v_2(z, t) \leq 1$ for all $(z, t) \in \mathbb{R} \times \mathbb{R}_+$, together with $v_1(z, 0) \leq v_0(z) \leq v_2(z, 0)$ for all $z \in \mathbb{R}$ at $t = 0$.

Proof. To begin with, we look for some suitable continuously differentiable functions $q_i : \mathbb{R}_+ \rightarrow (0, \infty)$ and $\xi_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ with the limits $\xi_i(t) \rightarrow \xi_{\infty,i} \in \mathbb{R}$ and $q_i(t) \rightarrow 0$ as $t \rightarrow \infty$; $i = 1, 2$.

As both cases of weak L^∞ -sub- and -supersolutions, v_1 and v_2 , respectively, are similar, we treat only the former one, the subsolution to eq. (2.1),

$$(3.33) \quad v_1(z, t) = v(z, t) = \max\{U(z - \xi(t)) - q(t), 0\}.$$

We leave out the index $i = 1$ in $\xi_1(t) = \xi(t)$ and $q_1(t) = q(t)$. If no confusion is likely, we also suppress the argument in $U = U(z - \xi(t))$.

Recalling Hypothesis (H5) on v_0 , inequalities (3.28), we first choose the constant $\eta \in (0, \eta_0]$ small enough, such that also inequalities (3.29) are valid; hence, $s_0 + 2\eta < v_0(+\infty)$. Of course, also (3.24) holds with η in place of η_0 . Then we choose $q_0 \in \mathbb{R}$ such that

$$s_0 + 2\eta < 1 - q_0 < \min\{v_0(+\infty), 1 - \eta\},$$

that is,

$$(3.34) \quad \max\{1 - v_0(+\infty), \eta\} < q_0 < 1 - s_0 - 2\eta.$$

Hence, there is some $z' > 0$ large enough, such that $U(z) - q_0 \leq 1 - q_0 \leq v_0(z + z')$ for all $z \geq 0$, which yields $U(z - z') - q_0 \leq v_0(z)$ for all $z \geq z'$. Since $\lim_{z \rightarrow -\infty} U(z) = 0$, there is another $z'' > 0$ large enough, such that $U(z - z'') - q_0 \leq 0 \leq v_0(z)$ for all $z \leq z'$. Setting $z^* = \max\{z', z''\} > 0$ and using the fact that $U : \mathbb{R} \rightarrow [0, 1]$ is monotone increasing, we arrive at $U(z - z^*) - q_0 \leq v_0(z)$ for all $z \in \mathbb{R}$. This inequality shows that the initial condition for the subsolution $v(z, t)$ is satisfied provided $\xi(0) \geq z^*$.

Now let us take the corresponding numbers $\delta \in (0, \eta)$ and $\underline{\mu} > 0, \bar{\mu} > 0$ in Hypothesis (H4), all of them depending on η fixed above. We need to distinguish among the following

three cases, $0 \leq U \leq \delta$, $\delta \leq U \leq 1 - \delta$, and $1 - \delta \leq U \leq 1$. We write the underlying domain $\mathbb{R} \times \mathbb{R}_+$ as the union $\mathbb{R} \times \mathbb{R}_+ = \Omega_\delta^{(1)} \cup \Omega_\delta^{(2)} \cup \Omega_\delta^{(3)}$ of the corresponding subsets

$$(3.35) \quad \Omega_\delta^{(1)} \stackrel{\text{def}}{=} \{(z, t) \in \mathbb{R} \times \mathbb{R}_+ : 0 \leq U(z - \xi(t)) \leq \delta\} \subset \mathbb{R} \times \mathbb{R}_+,$$

$$(3.36) \quad \Omega_\delta^{(2)} \stackrel{\text{def}}{=} \{(z, t) \in \mathbb{R} \times \mathbb{R}_+ : \delta \leq U(z - \xi(t)) \leq 1 - \delta\} \subset \mathbb{R} \times \mathbb{R}_+, \quad \text{and}$$

$$(3.37) \quad \Omega_\delta^{(3)} \stackrel{\text{def}}{=} \{(z, t) \in \mathbb{R} \times \mathbb{R}_+ : 1 - \delta \leq U(z - \xi(t)) \leq 1\} \subset \mathbb{R} \times \mathbb{R}_+.$$

We begin with the third case, $1 - \delta \leq U \leq 1$: Thanks to our choice of the function $q : \mathbb{R}_+ \rightarrow \mathbb{R}$, the inequalities $0 < q(t) \leq q_0$ hold for all $t \geq 0$. Hence, by ineq. (3.34), we have also

$$U(z - \xi(t)) - q(t) \geq U - q_0 \geq (1 - \delta) - q_0 > (1 - \delta) - (1 - s_0 - 2\delta) = s_0 + \delta > 0$$

which shows that the subsolution $v_1 = v$ in eq. (3.33) is given by

$$(3.38) \quad v(z, t) = U(z - \xi(t)) - q(t) \quad \text{for all } (z, t) \in \Omega_\delta^{(3)}.$$

It remains to verify the inequality

$$(3.39) \quad \mathcal{N}(v) \stackrel{\text{def}}{=} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial z^2} - c \frac{\partial v}{\partial z} - f(v) \leq 0, \quad (z, t) \in \Omega_\delta^{(3)},$$

which has to hold for a subsolution. Using (2.16) and (3.38) we calculate

$$(3.40) \quad \frac{\partial v}{\partial t} = -U'(z - \xi(t)) \xi'(t) - q'(t) = -V(U) \xi'(t) - q'(t),$$

$$(3.41) \quad \frac{\partial v}{\partial z} = U'(z - \xi(t)) = V(U),$$

$$(3.42) \quad \begin{aligned} & f(v(z, t)) - f(U(z - \xi(t))) \\ &= \Phi(U(z - \xi(t)), q(t)) \cdot q(t) = \Phi(U, q(t)) \cdot q(t), \quad \text{where} \end{aligned}$$

$$(3.43) \quad \Phi(s, q) \stackrel{\text{def}}{=} \frac{f(s - q) - f(s)}{q} \quad \text{for } s \in \mathbb{R}, \quad q \in \mathbb{R} \setminus \{0\}.$$

Consequently, the expressions in ineq. (3.39) become

$$(3.44) \quad \begin{aligned} \mathcal{N}(v) &= -U'(z - \xi(t)) \xi'(t) - q'(t) - U''(z - \xi(t)) - c U'(z - \xi(t)) \\ &\quad - f(U(z - \xi(t))) - \Phi(U(z - \xi(t)), q(t)) q(t) \\ &= -U' \xi' - q' - \Phi(U, q) q = -q \left[V(U) \frac{\xi'(t)}{q(t)} + \frac{d}{dt} \ln q(t) + \Phi(U, q) \right]. \end{aligned}$$

Notice that our choice of q_0 in (3.34) guarantees that $0 < q_0 \leq 1 - s_0 - 2\eta$. Thus, we may apply ineq. (3.26) to conclude that

$$\Phi(s, q) \geq \bar{\mu} \geq \mu \stackrel{\text{def}}{=} \min\{\underline{\mu}, \bar{\mu}\} > 0$$

holds for all pairs (s, q) satisfying $1 - \delta \leq s \leq 1$ and $0 < q \leq q_0$. Recall that the constant $\underline{\mu} > 0$ has been introduced in ineq. (3.25). Consequently, using eq. (3.44), we will easily obtain ineq. (3.39), that is, $\mathcal{N}(v) \leq 0$ for all $(z, t) \in \Omega_\delta^{(3)}$, provided we choose the functions $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $q : \mathbb{R}_+ \rightarrow (0, \infty)$ as in (3.32), where $\xi_{\infty,1} = \xi_\infty \in \mathbb{R}$ and $\nu > 0$ are some constants, such that $\xi(0) = \xi_\infty - \nu q_0 \geq z^*$ (to be specified later when we treat the second case). From the derivatives

$$(3.45) \quad \frac{q'(t)}{q(t)} = \frac{d}{dt} \ln q(t) = -\mu (< 0) \quad \text{and} \quad \frac{\xi'(t)}{q(t)} = \mu\nu (> 0)$$

inserted into the last bracket in eq. (3.44), we deduce the desired ineq. (3.39).

We continue with the first case, $0 \leq U \leq \delta$: While treating the third case above ($1 - \delta \leq U \leq 1$), we have chosen $q_0 \in \mathbb{R}$ such that inequalities (3.34) be satisfied. In analogy with the third case, we wish to show that $\mathcal{N}(v) \leq 0$ for all $(z, t) \in \Omega_\delta^{(1)}$. We write the set $\Omega_\delta^{(1)} = \Omega_\delta^{(1,+)} \cup \Omega_\delta^{(1,-)}$ as the union of the subsets

$$(3.46) \quad \Omega_\delta^{(1,+)} \stackrel{\text{def}}{=} \{(z, t) \in \Omega_\delta^{(1)} : q(t) \leq U(z - \xi(t))\} \quad \text{and}$$

$$(3.47) \quad \Omega_\delta^{(1,-)} \stackrel{\text{def}}{=} \{(z, t) \in \Omega_\delta^{(1)} : U(z - \xi(t)) < q(t)\}.$$

The subsolution $v_1 = v$ in eq. (3.33) is now given by

$$(3.48) \quad v(z, t) = \begin{cases} U(z - \xi(t)) - q(t) & \text{for } (z, t) \in \Omega_\delta^{(1,+)} , \\ 0 & \text{for } (z, t) \in \Omega_\delta^{(1,-)} . \end{cases}$$

Trivially, we have $\mathcal{N}(0) = -f(0) = 0$. It suffices to show that $\mathcal{N}(v) \leq 0$ holds for all $(z, t) \in \Omega_\delta^{(1,+)}$. This time we apply ineq. (3.25) with $\delta \in (0, \eta)$ and $\underline{\mu} > 0$ to conclude that

$$\Phi(s, q) \geq \underline{\mu} \geq \mu = \min\{\underline{\mu}, \bar{\mu}\} > 0$$

holds for all pairs (s, q) satisfying $0 < q \leq s \leq \delta$. We recall the inequalities $0 < \delta < s_0 - 2\eta$, by (3.24) and $0 < \delta < \eta \leq \eta_0$. Consequently, using eq. (3.44), we finally obtain $\mathcal{N}(v) \leq 0$ for all $(z, t) \in \Omega_\delta^{(1,+)}$ in the same way as in the third case above ($1 - \delta \leq U \leq 1$).

It remains to treat the second case, $\delta \leq U \leq 1 - \delta$: Here, we take advantage of the fact that there is a constant $\omega \in (0, \infty)$ such that $V(s) \geq \omega$ for all $s \in [\delta, 1 - \delta]$, by ineq. (2.16). Furthermore, ineq. (3.18) (a one-sided Lipschitz condition) guarantees that $\Phi(s, q) \geq -L$ whenever $0 < q \leq s \leq 1$. Similarly to the first case above ($0 \leq U \leq \delta$), we write the set $\Omega_\delta^{(2)} = \Omega_\delta^{(2,+)} \cup \Omega_\delta^{(2,-)}$ as the union of the subsets

$$(3.49) \quad \Omega_\delta^{(2,+)} \stackrel{\text{def}}{=} \{(z, t) \in \Omega_\delta^{(2)} : q(t) \leq U(z - \xi(t))\} \quad \text{and}$$

$$(3.50) \quad \Omega_\delta^{(2,-)} \stackrel{\text{def}}{=} \{(z, t) \in \Omega_\delta^{(2)} : U(z - \xi(t)) < q(t)\}.$$

The subsolution $v_1 = v$ in eq. (3.33) is given by

$$(3.51) \quad v(z, t) = \begin{cases} U(z - \xi(t)) - q(t) & \text{for } (z, t) \in \Omega_\delta^{(2,+)} , \\ 0 & \text{for } (z, t) \in \Omega_\delta^{(2,-)} . \end{cases}$$

Again, as in the first case, it suffices to show that $\mathcal{N}(v) \leq 0$ holds for all $(z, t) \in \Omega_\delta^{(2,+)}$. This time we apply eq. (3.45) and the inequalities

$$\begin{aligned} V(s) &\geq \omega \equiv \text{const} > 0 && \text{for all } s \in [\delta, 1 - \delta], \\ \Phi(s, q) &\geq -L \equiv \text{const} \leq 0 && \text{whenever } 0 < q \leq s \leq 1, \end{aligned}$$

obtained above to eq. (3.44), thus arriving at

$$\begin{aligned} \mathcal{N}(v) &= -q \left[V(U) \frac{\xi'(t)}{q(t)} + \frac{d}{dt} \ln q(t) + \Phi(U, q) \right] \\ &\leq -q [\omega \mu \nu - \mu - L] \leq 0 \quad \text{for all } (z, t) \in \Omega_\delta^{(2,+)}, \end{aligned}$$

provided $\nu > 0$ is chosen sufficiently large, such that $[\dots] = \omega \mu \nu - \mu - L \geq 0$. For instance,

$$\nu = \frac{1 + (L/\mu)}{\omega} > 0$$

will do it. Moreover, the remaining unspecified constant ξ_∞ from the third case above may be chosen to be

$$\xi_\infty = \nu q_0 + z^* = \frac{1 + (L/\mu)}{\omega} \cdot q_0 + z^* \in \mathbb{R}.$$

We have verified that $v_1 = v$ given by eq. (3.51) is a subsolution to the Cauchy problem (2.1) also in the second case ($\delta \leq U \leq 1 - \delta$).

This proves that the function $v_1 = v$ defined in eq. (3.33) is a subsolution to the Cauchy problem (2.1) in all three cases above. Moreover, v is Lipschitz-continuous in each set $\Omega_\delta^{(j)}$; $j = 1, 2, 3$, and, consequently, by its definition in (3.30), also in the whole space-time domain $\mathbb{R} \times \mathbb{R}_+ = \Omega_\delta^{(1)} \cup \Omega_\delta^{(2)} \cup \Omega_\delta^{(3)}$. It follows that v is a weak L^∞ -subsolution to (2.1).

A weak L^∞ -supersolution v_2 to the Cauchy problem (2.1) is obtained analogously, cf. eq. (3.31); we leave the details to an interested reader. We remark that the constant $\mu = \min\{\underline{\mu}, \bar{\mu}\} > 0$ remains the same for both v_i ; $i = 1, 2$. ■

4 Long-Time Asymptotic Behavior of Solutions

This section is concerned with the long-time asymptotic behavior of solutions $v = v(z, t)$ to the initial value problem (2.1) as time $t \rightarrow \infty$. More precisely, the uniform convergence of the family of functions $v(\cdot, t) : \mathbb{R} \rightarrow [0, 1]$; $t \in \mathbb{R}_+$, to a TW-solution $z \mapsto U(z + \zeta) : \mathbb{R} \rightarrow [0, 1]$ as $t \rightarrow \infty$ will be proved in our main result, Theorem 4.6. Our approach is based on controlling the long-time asymptotic behavior of $v(z, t)$ pointwise by a special pair of sub- and supersolutions constructed in Proposition 3.5. We combine the Lyapunov stability of TW-solutions with the spatial regularity of solutions $v(z, t)$ and the minimization of the Lyapunov functional on the ω -limit set of a solution as time goes to infinity in order to prove the long-time convergence in Theorem 4.6.

4.1 Lyapunov Stability of Travelling Waves

In this paragraph we establish the stability of travelling waves (equivalently, that of TW-solutions) in the sense of Lyapunov. These travelling waves for problem (1.1), $(x, t) \mapsto u(x, t) = U(x - ct + \zeta)$, where $\zeta \in \mathbb{R}$ is arbitrary, have been obtained in Section 2; see Proposition 2.3. Roughly speaking, if the (bounded classical) solution of problem (2.1), say, $v(\cdot, t) : \mathbb{R} \rightarrow [0, 1]$; $t \in \mathbb{R}_+$, has approached a TW-solution $z \mapsto U(z + \zeta)$ “close enough” at some time $t_0 \in \mathbb{R}_+$ and for some $\zeta \in \mathbb{R}$, that is to say, if the difference $v(z, t_0) - U(z + \zeta)$ is “small enough”, uniformly for all $z \in \mathbb{R}$, then the difference $v(z, t) - U(z + \zeta)$ stays “small” for all later times $t \geq t_0$, uniformly for all $z \in \mathbb{R}$. This claim, called *Lyapunov stability*, is quantified below.

Proposition 4.1 *Assume that f satisfies all four Hypotheses, (H1) through (H4). Let $z \mapsto U(z + \zeta) : \mathbb{R} \rightarrow [0, 1]$, $\zeta \in \mathbb{R}$, be a TW-solution as specified in Proposition 2.3, Part (d). Then there is a function $\varrho = \varrho(\varepsilon) > 0$ defined for every $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 > 0$ small enough, $\varepsilon_0 \leq \frac{1}{2} \cdot \min\{s_0, 1 - s_0\}$, such that $\lim_{\varepsilon \rightarrow 0+} \varrho(\varepsilon) = 0$ and the following property holds:*

(LS) *Let $\varepsilon \in (0, \varepsilon_0)$ be arbitrary. If $v_0 \in L^\infty(\mathbb{R})$ satisfies*

$$(4.1) \quad 0 \leq v_0(z) \leq 1 \quad \text{and} \quad |v_0(z) - U(z + \zeta)| \leq \varepsilon \quad \text{for all } z \in \mathbb{R},$$

where $\zeta \in \mathbb{R}$ is a suitable constant, then the (unique bounded classical) solution $v : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ of problem (2.1) (cf. Proposition 3.3) satisfies

$$(4.2) \quad |v(z, t) - U(z + \zeta)| \leq \varrho(\varepsilon) \quad \text{for all } (z, t) \in \mathbb{R} \times \mathbb{R}_+.$$

Since $v_0 \in L^\infty(\mathbb{R})$ in Condition (LS) satisfies inequalities (4.1) with $0 < \varepsilon \leq \varepsilon_0 \leq \frac{1}{2} \cdot \min\{s_0, 1 - s_0\}$, it obeys also Hypothesis (H5), with a help from (1.2).

Proof. Set $\varepsilon'_0 = \frac{1}{2} \cdot \min\{s_0, 1 - s_0\} > 0$. We begin with an arbitrary number $\varepsilon \in (0, \varepsilon'_0]$. Let $v_0 \in L^\infty(\mathbb{R})$ satisfy inequalities (4.1) with some constant $\zeta \in \mathbb{R}$. Next, we recall that, by Proposition 3.5, there is an ordered pair of weak L^∞ -sub- and -supersolutions of the Cauchy problem (2.1), $v_1(z, t)$ and $v_2(z, t)$, respectively, given by formulas (3.30) and (3.31), such that $0 \leq v_1(z, t) \leq v_2(z, t) \leq 1$ for all $(z, t) \in \mathbb{R} \times \mathbb{R}_+$, and $v_1(z, 0) \leq v_0(z) \leq v_2(z, 0)$ for all $z \in \mathbb{R}$. With regard to inequalities (4.1), we are going to find the constants $\mu, \nu, q_{0,i} \in (0, \infty)$ and $\xi_{\infty,i} \in \mathbb{R}$ in Proposition 3.5 ($i = 1, 2$), such that we have also

$$(4.3) \quad \begin{aligned} v_1(z, 0) &\leq \max\{U(z + \zeta) - \varepsilon, 0\} \leq v_0(z) \\ &\leq \min\{U(z + \zeta) + \varepsilon, 1\} \leq v_2(z, 0) \quad \text{for } z \in \mathbb{R} \end{aligned}$$

(at $t = 0$). Recalling eqs. (3.32) we observe that inequalities (4.3) above are satisfied provided the following two inequalities hold for every $z \in \mathbb{R}$:

$$(4.4) \quad U(z - \xi_{\infty,1}) - q_{0,1} \leq \max\{U(z + \zeta) - \varepsilon, 0\} \quad \text{and}$$

$$(4.5) \quad \min\{U(z + \zeta) + \varepsilon, 1\} \leq U(z + \xi_{\infty,2}) + q_{0,2}.$$

Clearly, by our choice of $\varepsilon \in (0, \varepsilon'_0]$, both functions of $z \in \mathbb{R}$,

$$(4.6) \quad v_{0,1}(z) = \max\{U(z + \zeta) - \varepsilon, 0\} \quad \text{and} \quad v_{0,2}(z) = \min\{U(z + \zeta) + \varepsilon, 1\},$$

satisfy Hypothesis **(H5)** for the initial condition v_0 , in addition to $v_{0,1} \leq v_0 \leq v_{0,2}$ on \mathbb{R} . Consequently, we may apply Proposition 3.5 to find some constants $\mu, \nu, q_{0,i} \in (0, \infty)$ and $\xi_{\infty,i} \in \mathbb{R}$ ($i = 1, 2$), such that both inequalities (4.4) and (4.5) are valid, provided ε is sufficiently small, say, $0 < \varepsilon \leq \varepsilon''_0 < \infty$. Setting $\varepsilon_0 = \min\{\varepsilon'_0, \varepsilon''_0\} > 0$ and inspecting our proof of Proposition 3.5, we conclude that the constants $\mu, \nu \in (0, \infty)$ can be chosen independent from $\varepsilon \in (0, \varepsilon_0]$; $\mu > 0$ sufficiently small and $\nu > 0$ sufficiently large. In contrast, $q_{0,i} \in (0, \infty)$ and $\xi_{\infty,i} \in \mathbb{R}$ depend on $\varepsilon \in (0, \varepsilon_0]$ in such a way that the estimates

$$(4.7) \quad 0 < q_{0,i} \equiv q_{0,i}(\varepsilon) \leq C\varepsilon; \quad i = 1, 2, \quad \text{and}$$

$$(4.8) \quad |\xi_{\infty,1} + \zeta| \leq C\varepsilon, \quad |\xi_{\infty,2} - \zeta| \leq C\varepsilon$$

hold with a constant $C > 0$ independent from $\varepsilon \in (0, \varepsilon_0]$. From the formulas in eqs. (3.32) we derive also

$$(4.9) \quad 0 < q_i(t) \leq q_{0,i} \leq C\varepsilon; \quad i = 1, 2,$$

$$(4.10) \quad |\xi_1(t) + \zeta| \leq |\xi_{\infty,1} + \zeta| + \nu q_1(t) \leq (1 + \nu)C\varepsilon, \quad \text{and}$$

$$(4.11) \quad |\xi_2(t) - \zeta| \leq |\xi_{\infty,2} - \zeta| + \nu q_2(t) \leq (1 + \nu)C\varepsilon$$

for all $t \in \mathbb{R}_+$ and for every $\varepsilon \in (0, \varepsilon_0]$. We combine these inequalities with the derivative $U' : \mathbb{R} \rightarrow \mathbb{R}$ being bounded to conclude that, by formulas (3.30) and (3.31), there is a constant $C' > 0$ independent from $\varepsilon \in (0, \varepsilon_0]$, such that

$$(4.12) \quad 0 \leq v_2(z, t) - v_1(z, t) \leq C'\varepsilon \quad \text{holds for all } (z, t) \in \mathbb{R} \times \mathbb{R}_+.$$

Since also all inequalities $v_1(z, t) \leq v(z, t) \leq v_2(z, t)$ and $v_1(z, t) \leq U(z + \zeta) \leq v_2(z, t)$ are valid for all $(z, t) \in \mathbb{R} \times \mathbb{R}_+$, by Lemma 3.1 (weak comparison principle), we arrive at the desired estimate (4.2) with the function $\varrho(\varepsilon) = C'\varepsilon$ defined for every $\varepsilon \in (0, \varepsilon_0]$. ■

4.2 Compactness and Regularity of Solution Orbits

The regularity and relative compactness of orbits generated by the unique solutions of the initial value problem (2.1) are treated in the next two lemmas.

Lemma 4.2 *Let $T \in (0, \infty)$ and let $\alpha \in (0, 1)$ be the Hölder exponent for f from Hypothesis **(H2)**. Given any Lebesgue measurable initial data $v_0 : \mathbb{R} \rightarrow [0, 1]$, let $v : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ denote the unique bounded classical solution of problem (2.1). Let us denote by $v_\tau(z, t) \stackrel{\text{def}}{=} v(z, t + \tau)$, for $(z, t) \in \mathbb{R} \times [0, T]$, the time translation of v by τ time units, $\tau \in (0, \infty)$. Then, for any $\tau_0 \in (0, \infty)$ there exists a constant*

$C_{(0,T)}^\alpha \equiv C_{(0,T)}^\alpha(\tau_0) \in \mathbb{R}_+$ such that the estimate

$$(4.13) \quad \max \left\{ \left\| \frac{\partial v_\tau}{\partial z} \right\|_{C^{\alpha,\alpha/2}(\mathbb{R} \times [0,T])}, \left\| \frac{\partial^2 v_\tau}{\partial z^2} \right\|_{C^{\alpha,\alpha/2}(\mathbb{R} \times [0,T])}, \left\| \frac{\partial v_\tau}{\partial t} \right\|_{C^{\alpha,\alpha/2}(\mathbb{R} \times [0,T])} \right\} \leq C_{(0,T)}^\alpha$$

holds for every time translation $\tau \geq \tau_0 > 0$.

Proof. A proof of ineq. (4.13) follows directly from the regularity results in (3.16). ■

We combine Lemma 4.2 with Arzelà-Ascoli's compactness criterion to obtain the following result.

Lemma 4.3 *Let $T \in (0, \infty)$ and let $\alpha \in (0, 1)$ be the Hölder exponent for f from Hypothesis (H2). Assume that $J \subset \mathbb{R}$ is a compact interval, $\beta \in (0, \alpha)$, and $0 < \tau_0 < \infty$. Under the hypotheses of Lemma 4.2, the family of “translation” functions $v_\tau : J \times [0, T] \rightarrow \mathbb{R}$; $\tau \geq \tau_0$, is relatively compact in the Hölder space $C^{2+\beta, 1+(\beta/2)}(J \times [0, T])$.*

4.3 Convergence of Solutions to a Travelling Wave

From now on we treat solely the case $-\infty < z_0 < z_1 < \infty$ as opposed to the classical case $z_0 = -\infty$ and $z_1 = +\infty$ ($f : \mathbb{R} \rightarrow \mathbb{R}$ continuously differentiable) treated in P. C. FIFE and J. B. McLEOD [7, Sect. 1] and J. D. MURRAY [17], §13.3, pp. 444–449. In §2.3 we have given a simple example when $-\infty < z_0 < z_1 < \infty$ (Example 2.4).

Let us recall that we work with initial data $v_0 : \mathbb{R} \rightarrow \mathbb{R}$ satisfying Hypothesis (H5). Given these initial data, in Proposition 3.5 we have constructed an ordered pair of weak L^∞ -sub- and -supersolutions, v_1 and v_2 , respectively, such that $v_1(z, 0) \leq v_0(z) \leq v_2(z, 0)$ for all $z \in \mathbb{R}$ at $t = 0$, and $0 \leq v_1(z, t) \leq v_2(z, t) \leq 1$ for all $(z, t) \in \mathbb{R} \times \mathbb{R}_+$. Especially formulas in (3.32) are of importance. Recalling our definition of z_0, z_1 in (2.25), we choose a compact interval $J = [a, b] \subset \mathbb{R}$ such that

$$(4.14) \quad a + \xi_{\infty,2} < z_0 < z_1 < b - \xi_{\infty,1}.$$

Since $\xi_i(t) \leq \xi_{\infty,i}$ for $t \geq 0$; $i = 1, 2$, this choice of J guarantees

$$(4.15) \quad a + \xi_2(t) < z_0 < z_1 < b - \xi_1(t) \quad \text{for all } t \geq 0,$$

together with the limits

$$(4.16) \quad \lim_{t \rightarrow \infty} v_1(z, t) = \lim_{t \rightarrow \infty} v_2(z, t) = 0 \quad \text{uniformly for } z \in (-\infty, a], \quad \text{and}$$

$$(4.17) \quad \lim_{t \rightarrow \infty} v_1(z, t) = \lim_{t \rightarrow \infty} v_2(z, t) = 1 \quad \text{uniformly for } z \in [b, +\infty).$$

Lemma 4.4 *Let $T \in (0, \infty)$. Assume that f satisfies all four Hypotheses, **(H1)** through **(H4)**. Let $J \subset \mathbb{R}$ be the compact interval specified above, $\beta \in (0, \alpha)$, and $0 < \tau_0 < \infty$. Assume that the initial data $v_0 : \mathbb{R} \rightarrow \mathbb{R}$ satisfy Hypothesis **(H5)** and let $v : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ denote the unique bounded classical solution of problem (2.1). Finally, let $v_\tau : J \times [0, T] \rightarrow \mathbb{R}$ be the family of “translation” functions defined in Lemma 4.2 for $\tau \geq \tau_0$. Then every sequence $\{\tau_n\}_{n=1}^\infty \subset [\tau_0, \infty)$, $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$, contains a subsequence, denoted again by $\{\tau_n\}_{n=1}^\infty$, such that*

$$(4.18) \quad v_{\tau_n} \rightarrow v^* \quad \text{in } C^{2+\beta, 1+(\beta/2)}(J \times [0, T]) \quad \text{as } n \rightarrow \infty,$$

for some function $v^* \in C^{2+\beta, 1+(\beta/2)}(J \times [0, T])$.

(i) *In particular, we have $v^*(a, t) = 0$, $v^*(b, t) = 1$, and*

$$(4.19) \quad U(z - \xi_{\infty, 1}) \leq v^*(z, t) \leq U(z + \xi_{\infty, 2}) \quad \text{for all } (z, t) \in J \times [0, T].$$

(ii) *The limit function $v^* : J \times [0, T] \rightarrow \mathbb{R}$ satisfies eq. (2.1) in $J \times [0, T]$ in the classical sense (with all partial derivatives being continuous). Consequently, its natural extension \tilde{v} to $\mathbb{R} \times [0, T]$,*

$$\tilde{v}(z, t) = \begin{cases} v^*(z, t) & \text{if } z \in J = [a, b]; \\ 0 & \text{if } z \in (-\infty, a); \\ 1 & \text{if } z \in (b, \infty), \end{cases}$$

defined for all $(z, t) \in \mathbb{R} \times [0, T]$, is a classical solution to eq. (2.1) in $\mathbb{R} \times [0, T]$.

(iii) *We have the uniform convergence on all of $\mathbb{R} \times [0, T]$,*

$$(4.20) \quad \sup_{(z, t) \in \mathbb{R} \times [0, T]} |v(z, t + \tau_n) - \tilde{v}(z, t)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. The convergence result in (4.18) follows directly from Lemma 4.3 above.

Proof of (i). The inequalities in (4.19) are derived from the initial data v_0 satisfying $v_1(z, 0) \leq v_0(z) \leq v_2(z, 0)$ for all $z \in \mathbb{R}$ at $t = 0$, by Hypothesis **(H5)** combined with Proposition 3.5 and the weak comparison principle in Lemma 3.1; first, for each function v_{τ_n} ; $n = 1, 2, 3, \dots$,

$$v_1(z, t + \tau_n) \leq v_{\tau_n}(z, t) \leq v_2(z, t + \tau_n) \quad \text{for all } (z, t) \in J \times [0, T],$$

then for the limit function v^* as $n \rightarrow \infty$.

The boundary behavior of v^* , i.e., $v^*(a, t) = 0$, $v^*(b, t) = 1$, is obtained from (4.15) and (4.19). From this boundary behavior of v^* , (4.18), and (4.19) we derive also

$$\frac{\partial v^*}{\partial z}(a, t) = \frac{\partial v^*}{\partial z}(b, t) = 0 \quad \text{and} \quad \frac{\partial^2 v^*}{\partial z^2}(a, t) = \frac{\partial^2 v^*}{\partial z^2}(b, t) = 0.$$

Proof of (ii). Each function v_{τ_n} is a bounded classical solution of the equation in (2.1). The convergence result in (4.18) yields that so is the limit function v^* . The remaining claims for \tilde{v} follow from (i) and its proof.

Proof of (iii). The uniform convergence on $J \times [0, T]$ follows from (4.18). To verify it also on the complement

$$(\mathbb{R} \setminus J) \times [0, T] = ((-\infty, a) \times [0, T]) \cup ((b, +\infty) \times [0, T]),$$

let us recall our choice of the compact interval $J = [a, b] \subset \mathbb{R}$ such that (4.14) is valid. Then formulas (3.30) and (3.31) yield $v_1(z, 0) \leq v_0(z) \leq v_2(z, 0)$ for all $z \in \mathbb{R}$, and $v_1(z, t) \leq v(z, t) \leq v_2(z, t)$ for all $(z, t) \in \mathbb{R} \times \mathbb{R}_+$. Recalling

$v_{\tau_n}(z, t) \stackrel{\text{def}}{=} v(z, t + \tau_n)$ for $(z, t) \in \mathbb{R} \times [0, T]$, and letting $n \rightarrow \infty$, we have also

$$\begin{aligned} (4.21) \quad U(z - \xi_{\infty,1}) &= \lim_{n \rightarrow \infty} v_1(z, t + \tau_n) \leq \tilde{v}(z, t) \\ &\leq \lim_{n \rightarrow \infty} v_2(z, t + \tau_n) = U(z + \xi_{\infty,2}) \quad \text{for } (z, t) \in (\mathbb{R} \setminus J) \times [0, T]. \end{aligned}$$

We combine these estimates with $v_1 \leq v \leq v_2$ in $\mathbb{R} \times \mathbb{R}_+$, thus arriving at

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sup_{(z,t) \in (\mathbb{R} \setminus J) \times [0,T]} |v(z, t + \tau_n) - \tilde{v}(z, t)| \\ &\leq \sup_{z \in \mathbb{R} \setminus J} (U(z + \xi_{\infty,2}) - U(z - \xi_{\infty,1})) = 0, \end{aligned}$$

thanks to (4.14). This proves the uniform convergence on the complement $(\mathbb{R} \setminus J) \times [0, T]$. We combine it with the uniform convergence on $J \times [0, T]$ in (4.18) to derive (4.20) in Part (iii). ■

In our treatment of the dynamical system generated by the solutions of problem (2.1) we use standard terminology from J. K. HALE [9]. The trivialized analogue of Lemma 4.4 for $T = 0$ enables us to define the ω -limit set $\omega(v_0)$ of the solution v as follows: Given any $0 < \tau < \infty$, the set of functions

$$\mathcal{O}_\tau = \{v(\cdot, t) : t \geq \tau\}$$

has a compact closure $\overline{\mathcal{O}_\tau}$ in $C^2(J)$. We define

$$\omega(v_0) \stackrel{\text{def}}{=} \bigcap_{\tau > 0} \overline{\mathcal{O}_\tau},$$

which is a nonempty compact set in $C^2(J)$. It is important to notice that all conclusions of Lemma 4.4, Part (i), remain valid also for every function $w \in \omega(v_0)$, whence

$$(4.22) \quad w(a) = 0, \quad w(b) = 1, \quad \text{and} \quad \frac{dw}{dz}(a) = \frac{dw}{dz}(b) = 0.$$

It is well-known that the ω -limit set $\omega(v_0)$ is invariant under the mapping $v^*(\cdot, 0) \mapsto v^*(\cdot, t)$ for any $t \in [0, T]$; see [9]. Next, we find an ω -limit point $v^*(\cdot, 0) \in \omega(v_0)$ such that $v^*(\cdot, t) \equiv v^*(\cdot, 0)$ in J for all $t \in [0, T]$.

We define a Lyapunov functional on $\omega(v_0)$ as follows:

$$(4.23) \quad \mathcal{E}(w) \stackrel{\text{def}}{=} \int_a^b \left[\frac{1}{2} \left(\frac{dw}{dz} \right)^2 - F(w(z)) \right] e^{cz} dz \quad \text{for } w \in \omega(v_0).$$

This functional is continuous on the nonempty compact set $\omega(v_0)$ in $C^2(J)$. Combining the equation in (2.1) for v^* with the boundary behavior (4.22) of $v^*(\cdot, t) \in \omega(v_0)$, we arrive at

$$(4.24) \quad \frac{d}{dt} \mathcal{E}(v^*(\cdot, t)) = - \int_a^b \left(\frac{\partial v^*}{\partial t} \right)^2 e^{cz} dz \leq 0 \quad \text{for } t \in [0, T],$$

cf. P. C. FIFE and J. B. MCLEOD [7, p. 350], proof of Lemma 4.5, or J. K. HALE [9, p. 76], §4.3.1. If $w_0 \in \omega(v_0)$ is a minimizer for $\mathcal{E} : \omega(v_0) \rightarrow \mathbb{R}$ and we choose the sequence $\{\tau_n\}_{n=1}^\infty \subset [\tau_0, \infty)$ in Lemma 4.4 such that $v(\cdot, \tau_n) \rightarrow w_0$ in $C^2(J)$ as $n \rightarrow \infty$, then we have also

$$(4.25) \quad v_{\tau_n} \rightarrow v^* \quad \text{in } C^{2,1}(J \times [0, T]) \quad \text{as } n \rightarrow \infty,$$

by (4.18) in Lemma 4.4, and $v^*(\cdot, 0) = w_0 \in C^2(J)$. If the function $t \mapsto v^*(\cdot, t) : [0, T] \rightarrow \omega(v_0) \subset C^2(J)$ were not constant in time t , then eq. (4.24) would yield $\mathcal{E}(v^*(\cdot, T)) < \mathcal{E}(w_0)$. This contradicts our choice of $w_0 \in \omega(v_0)$ to be a minimizer for $\mathcal{E} : \omega(v_0) \rightarrow \mathbb{R}$. Hence, we have $v^*(\cdot, t) \equiv w_0$ in J for all $t \in [0, T]$. The limit function $v^*(z, t) \equiv w_0(z)$ satisfies the equation in (2.1) for v^* with $\partial v^* / \partial t \equiv 0$ in $J \times [0, T]$, that is, eq. (2.5) together with (2.10), by Proposition 2.1, where the open interval (z_0, z_1) has to be replaced by another open interval $(\tilde{z}_0, \tilde{z}_1) \subset \mathbb{R}$. We apply Proposition 2.3, Part (d), to specify the shape of w_0 : There is a number $\zeta \in \mathbb{R}$ such that $w_0(z) = U(z + \zeta)$ for all $z \in \mathbb{R}$, and $(\tilde{z}_0, \tilde{z}_1) = (z_0 - \zeta, z_1 - \zeta) \subset J = [a, b]$. Consequently, (4.25) reads

$$(4.26) \quad \|v(z, t + \tau_n) - U(z + \zeta)\|_{C^{2,1}(J \times [0, T])} \longrightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where the norm is taken for the function $(z, t) \mapsto v(z, t + \tau_n) - U(z + \zeta) : J \times [0, T] \rightarrow \mathbb{R}$. We would like to emphasize that the shift $\zeta \equiv \zeta(v_0)$ depends on the choice of the initial data v_0 and on our choice of the ω -limit point $w_0 \in \omega(v_0)$. In turn, these choices specify also the sequence (or subsequence of) $\{\tau_n\}_{n=1}^\infty \subset [\tau_0, \infty)$.

We collect the results of this section up to now in the following proposition:

Proposition 4.5 *Assume that f satisfies all four Hypotheses, (H1) through (H4). Let $J \subset \mathbb{R}$ be the compact interval specified above, and $0 < \tau_0 < \infty$. Assume that the initial data $v_0 : \mathbb{R} \rightarrow \mathbb{R}$ satisfy Hypothesis (H5) and let $v : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ denote the unique bounded classical solution of problem (2.1). Then there exists a sequence $\{\tau_n\}_{n=1}^\infty \subset [\tau_0, \infty)$, $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$ such that, for some number $\zeta \in \mathbb{R}$, we have*

$$(4.27) \quad \|v(z, t + \tau_n) - U(z + \zeta)\|_{C^{2,1}(J \times [0, T])} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Furthermore, we have

$$\sup_{(z, t) \in \mathbb{R} \times [0, T]} |v(z, t + \tau_n) - U(z + \zeta)| \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We combine this proposition with the Lyapunov stability of travelling waves (Proposition 4.1) to prove our main result:

Theorem 4.6 (Convergence.) *Assume that the reaction function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies all four Hypotheses, (H1) through (H4), and the initial data $v_0 : \mathbb{R} \rightarrow \mathbb{R}$ satisfy Hypothesis (H5). Assume $-\infty < z_0 < z_1 < +\infty$ and let $J \subset \mathbb{R}$ be the compact interval specified above. Let $v : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ denote the unique bounded classical solution of problem (2.1). Then there is a spatial shift $\zeta = \zeta(v_0) \in \mathbb{R}$ uniquely determined by the initial data v_0 , such that*

$$(4.28) \quad \sup_{z \in \mathbb{R}} |v(z, t) - U(z + \zeta)| \longrightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Even the following stronger convergence in $C^2(J)$ holds (cf. (4.27)),

$$(4.29) \quad \|v(\cdot, t) - U(\cdot + \zeta)\|_{C^2(J)} \longrightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Proof. First, let $\zeta \in \mathbb{R}$ be the number obtained in Proposition 4.5 above. Next, we recall from Proposition 4.1 that the corresponding TW-solution $z \mapsto U(z + \zeta) : \mathbb{R} \rightarrow [0, 1]$ is Lyapunov-stable, i.e., (4.1) \implies (4.2). We now apply the conclusion of Proposition 4.5 as follows. For every $\varepsilon \in (0, \varepsilon_0]$ there exists an index $N = N(\varepsilon) \in \mathbb{N} = \{1, 2, 3, \dots\}$ such that, by (4.27), the following form of (4.1) is valid:

$$(4.30) \quad 0 \leq v(z, \tau_n) \leq 1 \quad \text{and} \quad |v(z, \tau_n) - U(z + \zeta)| \leq \varepsilon$$

for all $z \in \mathbb{R}$ and for every $n \geq N$.

Thus, by (4.2), we have

$$(4.31) \quad \sup_{z \in \mathbb{R}} |v(z, t + \tau_n) - U(z + \zeta)| \leq \varrho(\varepsilon) \quad \text{for all } t \in \mathbb{R}_+ \text{ and } n \geq N.$$

Consequently, given any $\varepsilon \in (0, \varepsilon_0]$, there exists some $t_0 = t_0(\varepsilon) \in [\tau_0, \infty)$, say, $t_0 = \tau_N$, such that

$$\sup_{z \in \mathbb{R}} |v(z, t) - U(z + \zeta)| \leq \varrho(\varepsilon) \quad \text{is valid for every } t \geq t_0.$$

Recalling $\varrho(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0+$, we infer from this convergence result that the spatial shift $\zeta \in \mathbb{R}$ is independent from the choice of the sequence $\{\tau_n\}_{n=1}^\infty \subset [\tau_0, \infty)$ in Proposition 4.5. It depends solely on the initial data v_0 . Moreover, we must have even (4.28), thanks to inequalities (4.30) and (4.31). The desired result (4.28) follows.

Finally, we apply the same regularity arguments as in the proofs of Lemma 4.4 and Proposition 4.5 to derive (4.29) from (4.28).

The theorem is proved. ■

Acknowledgments

The work of Pavel Drábek was supported in part by the Grant Agency of the Czech Republic (GAČR) under Grant #13 – 00863S, and the work of Peter Takáč by the Deutsche Forschungsgemeinschaft (DFG, Germany) under Grants # TA 213/15–1 and # TA 213/16–1. Both authors were partially supported also by a joint exchange program between the Czech Republic and Germany; by the Ministry of Education, Youth, and Sports of the Czech Republic under the grant No. 7AMB14DE005 (exchange program “MOBILITY”) and by the Federal Ministry of Education and Research of Germany under grant No. 57063847 (D.A.A.D. Program “PPP”). Both authors would like to express their sincere thanks to Professor Hiroshi Matano (University of Tokyo, Japan) for suggesting to them the monotonicity of travelling waves in Proposition 2.1.

References

1. D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Advances in Math., **30** (1978), 33–76.
2. J. M. Ball, *Strongly continuous semigroups, weak solutions, and the variation-of-constants formula*, Proc. Amer. Math. Soc., **63**(2) (1977), 70–73.
3. Klaus Deimling, *“Nonlinear Functional Analysis”*, Springer-Verlag, Berlin-Heidelberg, 1985.
4. P. Drábek, R. F. Manásevich, and P. Takáč, *Manifolds of critical points in a quasi-linear model for phase transitions*. In D. Bonheure, M. Cuesta, E. J. Lami Dozo, P. Takáč, J. Van Schaftingen, and M. Willem; eds., *“Nonlinear Elliptic Partial Differential Equations”*, Proceedings of the 2009 “International Workshop in Nonlinear Elliptic PDEs,” A celebration of Jean-Pierre Gossez’s 65-th birthday, September 2–4, 2009, Brussels, Belgium. Contemporary Mathematics, Vol. **540**, pp. 95–134, American Mathematical Society, Providence, R.I., U.S.A., 2011.
5. P. Drábek and J. Milota, *“Methods of Nonlinear Analysis”*, 2nd ed., in *Birkhäuser Advanced Texts*. Springer-Verlag, Basel-Heidelberg-New York, 2013.
6. P. Drábek and P. Takáč, *New patterns of travelling waves in the generalized Fisher-Kolmogorov equation*, Nonlinear Differ. Equ. Appl. (NoDEA), **23**(2) (2016), Article 7 (online). *Online:* <http://dx.doi.org/10.1007/s00030-016-0365-2>.
7. P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Rational Mech. Anal., **65**(4) (1977), 335–361.
8. R. A. Fisher, *The advance of advantageous genes*, Ann. of Eugenics, **7** (1937), 355–369.
9. Hale, J. K., *“Asymptotic Behavior of Dissipative Systems”*, Math. Surveys and Monographs, Vol. **25**. American Math. Soc., Providence, R.I., 1988.

10. F. Hamel and N. Nadirashvili, *Travelling fronts and entire solutions of the Fisher-KPP equation in \mathbb{R}^N* , Arch. Rational Mech. Anal., **157** (2001), 91–163.
11. D. Henry, “*Geometric Theory of Semilinear Parabolic Equations*”, in *Lect. Notes in Math.*, Vol. **840**. Springer-Verlag, Berlin-Heidelberg-New York, 1981.
12. A. Kolmogorov, I. Petrovski, and N. Piscounov, *Étude de l'équation de la diffusion avec croissance de la quantité de la matière et son application à un problème biologique*, Bull. Univ. Moskou Ser. Internat. Sec. A, **1** (1937), 1–25.
13. O. A. Ladyzhenskaya, N. N. Ural'tseva, and V. A. Solonnikov, “*Linear and Quasi-linear Equations of Parabolic Type*”. In *Transl. Mathematical Monographs*, Vol. **23**, Amer. Math. Soc., Providence, R.I., 1968.
14. H. Matano and T. Ogiwara, *Stability in order-preserving systems in the presence of symmetry*, Proc. Royal Soc. Edinburgh, **129 A** (1999), 395–438.
15. H. Matano and T. Ogiwara, *Monotonicity and convergence results in order-preserving systems in the presence of symmetry*, Discrete and Continuous Dynamical Systems, **5**(1) (1999), 1–34.
16. J. D. Murray, “*Mathematical Biology*”, in *Biomathematics Texts*, Vol. **19**, Springer-Verlag, Berlin–Heidelberg-New York, 1993.
17. J. D. Murray, “*Mathematical Biology I: An Introduction*”, 3-rd Ed. In *Interdisciplinary Applied Mathematics*, Vol. **17**, Springer-Verlag, Berlin–Heidelberg-New York, 2002.
18. A. Pazy, “*Semigroups of Linear Operators and Applications to Partial Differential Equations*”, in *Applied Mathematical Sciences*, Vol. **44**, Springer-Verlag, New York–Berlin–Heidelberg, 1983.
19. A. Tsoularis and J. Wallace, *Analysis of logistics growth models*, Math. Biosciences, **179**(1) (2002), 21–55.